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NEWS AND LETTERS

COVER: A postage stamp issued in India on December 22, 1967, the seventy-fifth birthday of Srinivasa Ramanujan whose notebooks are discussed on pp. 147–164.

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ABOUT OUR AUTHORS

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Ramanujan's Notebooks

Working mostly in isolation, Ramanujan noted striking and sometimes still unproved results in series, special functions and number theory.

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We first perceive, then reason later.
Vivekananda

His life spanned but 33 years, and most of these years were lived in virtual obscurity. However, out of such humble beginnings, rose Srinivasa Ramanujan, India's most famous mathematician.

The first purpose of this paper is to give a short account of the life of Ramanujan. While he was in the Indian equivalent of high school, Ramanujan commenced the recording of his mathematical discoveries in notebooks. He continued this practice in the penniless and jobless years which followed. Our second task then is to present a detailed history of Ramanujan's Notebooks from their inception to the time of their publication in 1957. Thirdly, we shall delineate the contents of the Notebooks and describe some of the more fascinating formulas found therein.

Several of the theorems in the Notebooks are rediscoveries by Ramanujan of earlier theorems of others. Some of the results found in the Notebooks were eventually published by Ramanujan, and after his death others, in particular Hardy and Watson, gave proofs of some of the theorems in the Notebooks. In the decade following Ramanujan's death, several papers were published in the Journal of the Indian Mathematical Society that proved results in Ramanujan's Notebooks. However, for the most part, the authors of these papers were unaware that their results were buried in the Notebooks. In particular, the papers of Chowla **[13]**, Malurkar **[25]**, and Rao and Aiyar **[43]** should be mentioned. It would be an impossible task to reference all material in the Notebooks that is proved elsewhere. However, when known to the author, references to other work will be given. Nonetheless, to this day, many of the discoveries in the Notebooks remain unverified. It is hoped that this paper might stimulate others to investigate this treasure of wondrous formulae and inspirational creativity.

Fuller accounts of the life of Ramanujan may be found in the obituary notices of Aiyar and Rao **[1]**, **[44]**, **[40]** and Hardy **[15]**, **[16]**, in the lecture of Hardy **[20]**, in the reminiscences of Ranganathan **[41]**, in the Ramanujan Commemoration Volume **[9]**, and in the review of Mordell **[26]**. For a brief sketch of the early history and another description of the contents of the Notebooks, consult the lecture of Watson **[49]**. In addition to the Notebooks, Ramanujan left behind some unpublished manuscripts on the partition and Ramanujan tau functions. These manuscripts have recently been examined by Rankin [42] and Birch [10]. In 1976, George Andrews discovered a "lost notebook" of Ramanujan containing over 100 pages tucked away in the Cambridge University library. This manuscript undoubtedly contains much of the last work of Ramanujan done in the last year of his life.

Life History

Ramanujan was born on December 22, 1887, in Erode, a town in southern India. As was the custom in that time, he was born in the home of his maternal grandparents. Ramanujan's father was a petty accountant to cloth merchants in Kumbakonam about 120 miles east of Erode and about 160 miles south-southwest of Madras. Although he was born into a Brahmin family, his parents were rather poor but not destitute.

Ramanujan's precocity was perhaps first noticed at the age of twelve when he borrowed from an older student a copy of Loney's text on trigonometry. He completely mastered its contents and worked every problem in the book. At the age of sixteen, he borrowed Carr's *Synopsis of Pure Mathematics* from the library of the local Government College. This book was, no doubt, the most significant influence in Ramanujan's mathematical development. The book contains the statements of over 6,000 theorems. There are very few proofs in it, and those proofs that are given are only briefly sketched. The material predates about 1880. It is significant that the book contains no material on elliptic functions, a subject in which Ramanujan became an expert, and nothing on functions of a complex variable, a subject which Ramanujan evidently never learned. In recent years, a later edition of Carr's book [12] was reprinted. However, despite being a plentiful source of information, the book is little known and sparingly used today.

In December, 1903, Ramanujan took the matriculation exam of the University of Madras and obtained a "first class" place. However, due to his complete absorption into mathematics and his failure to study English and physiology, a subject which he disliked intensely, Ramanujan failed the exam at the end of his first year at the Government College in Kumbakonam. His parents did not wish their son to devote so much time to mathematics, and it is reported that he did much of his mathematics under a cot in an attempt to escape the eyes of his parents [41, p. 86]. He later entered Pachaiyappa's College in Madras but, in 1907, again failed his examination.

From 1907 to 1909 Ramanujan wandered about the countryside impoverished and unemployed, but he continued to do mathematics and record his results in his Notebooks. In 1909, Ramanujan married; his wife Srimathi Janaki recently received the first copy of a commemorative volume [9] dedicated to her late husband. In 1910, Ramanujan met V. R. Aiyar, the founder of the Indian Mathematical Society. At that time, Aiyar was a deputy collector in the Madras civil service, and Ramanujan asked him for a job in his office. After perusing the theorems in Ramanujan's Notebooks, Aiyar wrote P. V. Seshu Aiyar, Ramanujan's mathematics instructor while at the Government College in Kumbakonam. Seshu Aiyar, in turn, wrote to another mathematician R. Ramachandra Rao who arranged a meeting with Ramanujan. The contents of Ramanujan's Notebooks exceedingly excited Rao, and he gave Ramanujan a monthly allowance so that Ramanujan could continue his work on mathematics. Thus, for the first time, Ramanujan could devote his full energies to mathematics without the burden of worrying about money for daily subsistence. However, Ramanujan was bothered by the fact that he was receiving money without really formally occupying a job. Thus, on March 1, 1912, Ramanujan became a clerk in the Madras Port Trust Office. This turned out to be most fortuitous, for the chairman of the Madras Port Office was Sir Francis Spring who took a great interest in Ramanujan and his mathematical work.

Spring was a prominent engineer who knew a fair amount of mathematics, and the manager S. N. Aiyar of the Madras Port Office was a well-known mathematician. They, along with P. V. Seshu Aiyar, encouraged Ramanujan to write to the famed English mathematican G. H. Hardy, and on January 16, 1913, Ramanujan sent the first of his now famous letters to Hardy. Hardy [40, p. xxii] later indicated that he felt that Ramanujan had considerable help in drafting the letter because

Ramanujan's facility with English was lacking. Indeed, apparently the letter was composed, for the most part, by S. N. Aiyar and Sir Francis Spring [**9**, p. 47]. The letter and two subsequent letters to Hardy that followed shortly thereafter contained the statements of approximately 120 theorems. On receiving the first letter, Hardy dismissed the results contained therein as the work of a crank. However, on that evening, he and Littlewood examined the letter with considerably more care, and, despite the falsity of some of the results, they were astonished with many of the findings. Hardy wrote to Ramanujan with the strong suggestion that he come to Cambridge so that his already great talents could be further developed. Because of strong caste convictions and the refusal of his mother to give permission, Ramanujan declined Hardy's invitation. Further, partly through the persuasion of Sir Gilbert Walker, on March 19, 1913, Ramanujan was elected to a scholarship at Madras. Thus, on May 1, Ramanujan began an academic career that would allow him to devote his full resources to mathematics without going to England.

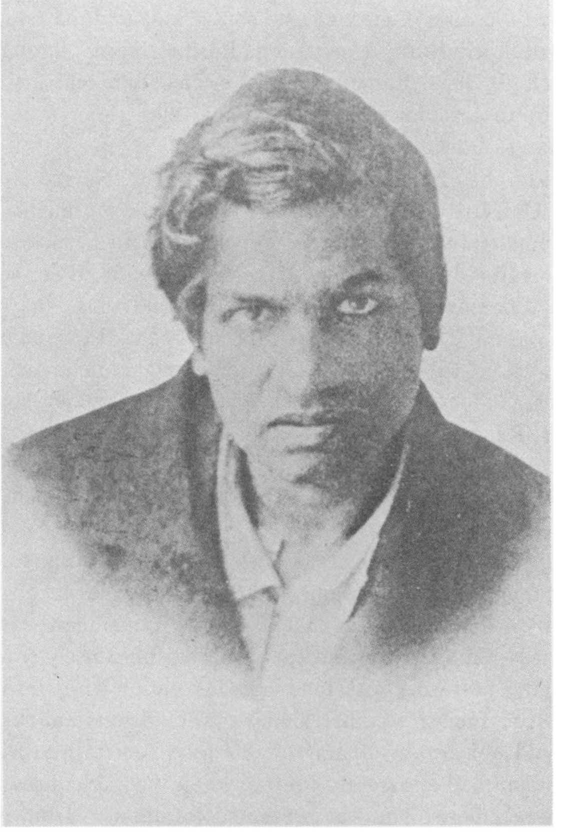
At the beginning of 1914, the Cambridge mathematician E. H. Neville sailed to India to lecture in the winter term at the University of Madras. One of Neville's principal tasks was to convince Ramanujan to come to England. Probably more important than the persuasions of Neville were a pilgrimage of Ramanujan and a dream of his mother. According to N. Subbanarayanan [**9**, p. 48], his father, S. N. Aiyar, and Ramanujan made a trip to Namakkal where they stayed in the temple of Goddess Namagiri three days and three nights. On the last night, Ramanujan received the command from Goddess Namagiri to go to England. His mother had a dream in which she saw her son surrounded by Europeans. Goddess Namagiri then told her to allow her son to depart for England. Thus, on March 17, 1914, Ramanujan left Madras for England. The departure was very difficult for Ramanujan. He was persuaded to cut off his tuft of hair which had religious significance to him. The unaccustomed wearing of European clothes and shoes made him quite uncomfortable and did not mitigate his anxiety.

On April 14, Ramanujan reached London. (Historical accounts [**9**, pp. 43, 72, 109], [**41**, pp. 70–71] differ as to whether Ramanujan was accompanied on the boat by Neville or whether Neville only met Ramanujan at the dock.) In the three years that followed, Ramanujan was very active mathematically. He talked with Hardy almost daily, and each profited immensely from the stimulation of the other. Much of Ramanujan's best work was done in collaboration with Hardy. However, Ramanujan found it difficult to adjust to English weather. Being a strict vegetarian, he cooked for himself. Due partly to the war, he could not always obtain suitable food sent to him from India. In May 1917, Hardy wrote that Ramanujan had apparently contracted an incurable disease which in the following year was diagnosed as tuberculosis. The war prevented Ramanujan from immediately returning to India, but on February 27, 1919, Ramanujan departed from England. The more favorable weather and diet of India, however, did not abate Ramanujan's illness, and on April 26, 1920, he died.

History and general content of Ramanujan's Notebooks

In 1903, or perhaps earlier, while in school, Ramanujan began to collect his theorems in notebooks. He added to his notebooks until about 1913. There are two main Notebooks. The second is a revision of the first and contains additional material as well. The second edition might have been started while Ramanujan held a scholarship at Madras. There is also a very short third Notebook which is of a fragmentary nature. Together, the three Notebooks contain the statements of approximately 3000–4000 theorems. Emil Grosswald has suggested the possibility that Ramanujan left us four notebooks in that the second Notebook, in reality, contains two separate notebooks. In the last 53 pages of what is commonly designated as Notebook 2, Ramanujan commenced a new page numbering system. The Notebooks which survive are probably not the earliest notebooks in which Ramanujan recorded his results. In 1912, Sir Gilbert Walker saw 4 or 5 notebooks of Ramanujan, each about an inch thick and each in a black cover.

The first of Ramanujan's Notebooks was left with Hardy when Ramanujan left England in 1919. The second and third Notebooks were taken back to India and were acquired by the University of Madras on Ramanujan's death. Apparently, in India, the location of the first Notebook was not

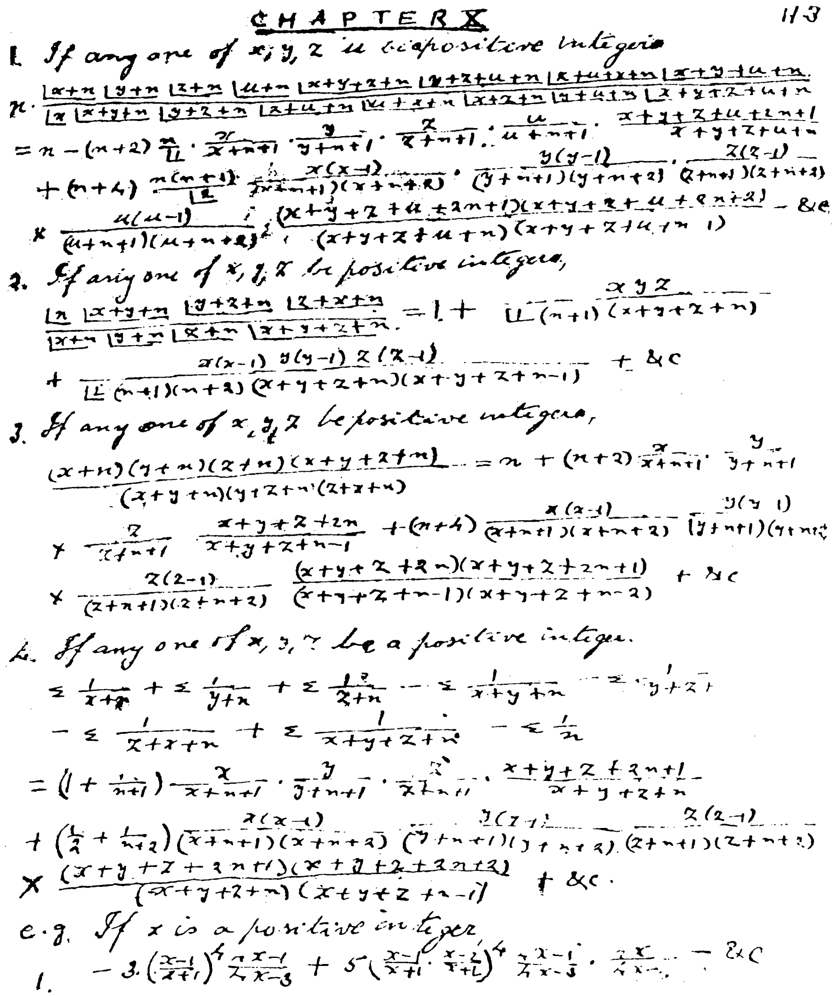


Srinivasa Ramanujan

known at that time. In 1924, S. R. Ranganathan, a mathematician and the University librarian at Madras, journeyed to Cambridge to study library science for a year. While in England, he visited Hardy who was now at Oxford. Hardy gave Ranganathan the first Notebook and strongly urged the publication and editing of the Notebooks. Ranganathan took the Notebook back to Madras where three handwritten copies of all three Notebooks were made. One of the handwritten copies was sent to Hardy.

With the assistance of B. M. Wilson, G. N. Watson undertook the task of editing the Notebooks in 1929. At that time, he estimated that it would take them at least five years to accomplish the task. Partly due to the premature death of Wilson, the task of editing the Notebooks was never completed. However, Watson wrote approximately 25 papers inspired by theorems in the Notebooks, especially the latter parts of the second Notebook. Thus, more than any other mathematician, Watson brought some of the contents before the mathematical public even though he never realized his initial aim of editing the entire Notebooks.

Late in the first half of this century, interest in publishing and editing the Notebooks was revived. In 1949, three photostat copies of the Notebooks were made at the University of Madras, and two of



A typical page from the second volume of Ramanujan's notebooks.

them still remain in the library there. The third copy was given to the noted Indian mathematician S. S. Pillai. In 1950, Pillai lost his life in a plane crash in Egypt while enroute to the International Congress of Mathematicians in Cambridge, Massachusetts. His copy of the Notebooks was with him.

In 1954, the publishing of the Notebooks was suggested at the meeting of the Indian Mathematical Society in Delhi. Finally, in 1957, the Tata Institute of Fundamental Research in Bombay published in two volumes a photostat copy of the Notebooks edited by K. Chandrasekharan. The first volume reproduces Ramanujan's first Notebook, while the second reproduces his second and third Notebooks. Only 1000 copies of the Notebooks were published. The facsimile edition contains no commentary whatsoever and is undoubtedly far less than the project originally envisioned by Hardy and Watson. The reproduction, however, is very faithful and clear. If one side of a page is left blank by Ramanujan in the Notebooks, it is left blank in the facsimile edition. Ramanujan's "scratch" work is also faithfully reproduced. Thus, on one page, we find only the fragment, "If r is positive." The reproduction was done on heavy, large pages with very generous margins. The author weighed the two volumes on his bathroom scales, and they weighed together over ten pounds. On the other hand, the excellent reproduction on such sturdy paper will insure their preservation for the years to come, and the mathematical community should be very grateful to the Tata Institute for performing such a valuable service.

The first of Ramanujan's Notebooks was written in a peculiar green ink. The book has 16 chapters containing 134 pages. Following these 16 chapters are approximately 80 pages of heterogeneous material. At first, Ramanujan wrote on only one side of the page. He later began using the back side of the pages for miscellaneous "scratch" work, which, for the most part, had no connection with the remainder of the material in the chapter. The chapters are somewhat organized into topics, but often there is no apparent connection between adjacent sections of material in the same chapter. Very seldom is there any indication of a proof or verification of a theorem.

The second Notebook contains 21 chapters with a total of 252 pages. These chapters are followed by 100 pages of heterogeneous formulas. In contrast to the first Notebook, Ramanujan used both sides of the paper in the second Notebook. Brief proofs are somewhat more frequent in the second Notebook than in the first, but by no means are they very numerous. Many people have conjectured about the nature of Ramanujan's proofs. His proofs were undoubtedly a mixture of intuition, examples, and induction, but evidently little more can be said. The successive listing of theorems with few verifications was clearly influenced by the style of Carr's *Synopsis*.

Chapters 1 and 2 of each volume cover, respectively, the same topics. Chapter 3 of the second Notebook contains material from Chapters 3 and 4 of the first. Chapter 4 of the second Notebook has material from Chapter 5 of the first. The material from Chapters 6 and 7 of the first is found in Chapter 5 of the second. Chapter n, $6 \leqslant n \leqslant 12$, of the second Notebook roughly corresponds to Chapter $n+2$ of the first, although material from Chapter 12 of the first Notebook can be found in both Chapters 10 and 11 of the second. Contents of Chapter 15 of the first are found in Chapters 16 and 17 of the second, and material from Chapter 16 of the first is in Chapter 14 of the second. Chapters 13, 15 and 18–21 of the second Notebook contain largely material not found in the first Notebook.

The third Notebook contains but 33 pages of miscellaneous work and is not divided into chapters. In this Notebook are found some of Ramanujan's results in number theory. There is an interesting story [**41**, p. 16] in connection with a table of natural numbers found on pages 8 and 9 of the third Notebook (pp. 368–369 of [**39**, vol. II]). In 1934, over a ouija board, Ranganathan and a friend of Ramanujan in his youth, K. S. Krishnaswami Ayyangar, invoked the spirit of Ramanujan. In replying to a question concerning the continuation of his mathematics, Ramanujan remarked that "All interest in mathematics dropped out after crossing over." Ramanujan was also queried about the meaning of the aforementioned table of numbers. He could not remember anything about the table. However, having been invoked in another seance one week later, Ramanujan informed that the table was related to his work on mock theta functions. This seems very unlikely. First, the work in the Notebooks was done before Ramanujan left for England, while the theory of mock theta functions was Ramanujan's last contribution to mathematics before he died. Secondly, the table consists of all natural numbers through 12,005 whose factorizations contain only the primes 2, 3, 5, and 7, and so has no connection with mock theta functions, but instead with Ramanujan's work on highly composite numbers [**37**].

Before giving some specific examples from the Notebooks, we shall make a few general remarks on their contents. Ramanujan clearly loved infinite series. In the formal manipulation of series, possibly only Euler and Jacobi were his equals. Many identities involving infinite series found in his Notebooks remain unverified to this day. Often the series do not converge, and seldom was there any mention of this. He discovered many methods of summability, including those of Abel and Borel, but Ramanujan made no distinction between ordinary convergence and summability. However, he did have a rather peculiar theory of divergent series. Ramanujan seldom used the summation sign Σ and instead preferred to write out the first few terms of the series followed by the symbol &c. Occasionally, he would write, for example, $\Sigma\ 1/n$ with no indices of summation. The only other instances of the employment of the summation sign are in double summations where the summation sign is used to denote the outer summation. At times, Ramanujan's infrequent discourse on series is obscure. For example, on page 181 of volume 2, Ramanujan states that "...$F(h)$ also will be an infinite series; but if most of the numbers p,q,r,s,t &c be odd integers $F(h)$ appears to terminate. In this case the hidden part of $F(h)$ can't be expanded in ascending powers of h..." It is not always clear why Ramanujan considered various functions defined by infinite series. Intriguing series identities are produced throughout the Notebooks. Often, changes of variables and the specialization of parameters yield beautiful results. There are many evaluations of series which are not found in any of the standard tables. Many of these identities and evaluations, as well as other theorems of Ramanujan would serve as interesting problems for the MONTHLY.

Several of Ramanujan's intriguing series identities resulted from adroit applications of variants of the Poisson summation formula and various other types of summation formulas. With suitable restrictions on f, the Poisson summation formula may be written as

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)\cos(2\pi n x)\,dx.$$

This, the Abel-Plana summation formula, and several new summation formulas were discovered by Ramanujan. He never stated conditions under which the formulas were valid. Various versions of the Euler-Maclaurin summation formula were skillfully used by Ramanujan in evaluating series, in obtaining series relations, and in approximating sums. For recent accounts of the Euler-Maclaurin formula and its applications, see the papers of Boas [**11**] and Berndt and Schoenfeld [**8**].

Ramanujan was also quite fond of the Bernoulli numbers B_n which are customarily defined today by

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n \qquad (|x| < 2\pi).$$

At least four different conventions for the Bernoulli numbers are found in the literature, and Ramanujan used at least two of them in the Notebooks; the reader must be careful about this. Many of the formulae that Ramanujan derived involve Bernoulli numbers. Ramanujan's first published paper [**30**], [**40**, pp. 1–14] is on Bernoulli numbers. Ramanujan calculated several of the Bernoulli numbers and had an excellent facility for calculation, although he was not as exceptional as many others.

Although many unusual integrals, several involving infinite products, are evaluated in the Notebooks, Ramanujan's contributions in this area are not nearly as numerous or as profound as his contributions to infinite series. Several beautiful identities between integrals arose from a skillful use of Parseval's identity for integrals or from similar theorems, all of which Ramanujan discovered independently.

A major portion of the latter half of Ramanujan's second Notebook is devoted to formulae from the theory of elliptic modular functions. In this area, Ramanujan is surpassed by no other. Watson [**49**] remarked that "A prolonged study of his modular equations has convinced me that he was in possession of a general formula by means of which modular equations can be constructed in almost terrifying numbers."

Ramanujan's continued fraction expansions of functions are striking. Among the theorems enounced in Ramanujan's first letter to Hardy, Hardy found those on continued fractions perhaps the most interesting.

Ramanujan used the old-fashioned notation $\underline{n|}$ for $n!$. He also used the notation $\underline{n|}$ when n is not a positive integer, where we, of course, would denote $\underline{n|}$ by the gamma function $\Gamma(n+1)$. As customary in his day, Ramanujan denoted the Riemann zeta-function $\zeta(n) = \sum_{k=1}^{\infty} k^{-n}$, $n > 1$, by S_n. The reader should be warned, however, that the notation S_n was used by Ramanujan in other contexts as well.

In a certain sense, mathematics has been advanced most by those who are distinguished more for intuition than for rigorous methods of proof.
—Felix Klein

Some fascinating entries from the Notebooks

Since the second Notebook is an expanded version of the first, we shall make most of our references to the second Notebook. The indicated pagination will always be that of the edition published by the Tata Institute. Formulas are quoted as Ramanujan wrote them, and no attempt has been made to verify them. For economy of space, we shall frequently use the summation sign Σ instead of Ramanujan's notation.

A magic square is a square array of natural numbers such that the sum of the elements in any column, row, or diagonal is the same. For several centuries, magic squares have been popular in elementary Indian mathematics. Ramanujan's very early preoccupation with mathematics focused on magic squares. In the opening chapter of each Notebook, Ramanujan gives details on how to construct magic squares and other square arrays satisfying certain properties. Details are much fuller in the second volume than in the first. In [**9**, p. 100], M. Venkataraman gives the following magic square:

22	12	18	87
21	84	32	2
92	16	7	24
4	27	82	26

constructed according to directions found in the Notebooks. Note that the first row gives the date of Ramanujan's birth. For excellent accounts of magic squares, consult the books of Ball and Coxeter [**5**] and W. S. Andrews [**3**].

In the second Notebook, Chapter 1 is prefaced by two pages of columns of natural numbers. The numbers in the columns at the right are highly composite numbers, i.e., numbers with many small prime factors. The columns at the left give the number of divisors of each of these highly composite numbers. One of Ramanujan's most significant papers [**37**], [**40**, pp. 78–128] is concerned with this topic.

Chapter 2 is largely devoted to sums involving the reciprocals of integers. On page 7 of the first Notebook we find the following interesting derivation. Ramanujan first proves that

$$\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}=\frac{n}{2n+1}+\frac{1}{2^3-2}+\frac{1}{4^3-4}+\cdots+\frac{1}{(2n)^3-2n}. \tag{1}$$

He then lets $n=1/dx$ in the left side of (1). From

$$\frac{2dx}{1+dx}+\frac{2dx}{1+2dx}+\cdots+\frac{2dx}{1+1}=2\int_1^2\frac{dx}{x}=2\log 2 \tag{2}$$

and (1), upon letting n tend to ∞, Ramanujan deduces that

$$1+\sum_{n=1}^{\infty}\frac{2}{(2n)^3-2n}=2\log 2.$$

Of course, the first equality in (2) is not quite correct. Ramanujan is really approximating the integral on the right by a Riemann sum. The first problem that Ramanujan published in the Journal of the Indian Mathematical Society is a very similar result [**28**], [**40**, p. 322].

Chapter 2 also contains some identities that are useful in obtaining approximations of partial sums of the harmonic series. Thus, on page 17 of the second Notebook, Ramanujan says that

$$\sum_{n=1}^{1000} \frac{1}{n} = 7\tfrac{1}{2} \text{ very nearly.}$$

Ramanujan was quite fond of using the expressions "nearly" and "very nearly".

Ramanujan had a proclivity for sums involving $\tan^{-1}x$, and in Chapter 2, we find several such finite sums. As an example, we quote (vol. II, p. 16)

$$\begin{aligned}
&\tan^{-1}\frac{1}{n+1} + \tan^{-1}\frac{1}{n+2} + \cdots + \tan^{-1}\frac{1}{2n} + \tan^{-1}\frac{1}{2n+1} + \tan^{-1}\frac{1}{2n+3} + \cdots + \tan^{-1}\frac{1}{4n+1} \\
&= \frac{\pi}{4} + \tan^{-1}\frac{9}{53} + \tan^{-1}\frac{18}{599} + \cdots + \tan^{-1}\frac{9n}{32n^4+22n^2-1} \\
&\quad + \tan^{-1}\frac{4}{137} + \tan^{-1}\frac{8}{2081} + \cdots + \tan^{-1}\frac{4n}{128n^4+8n^2+1}.
\end{aligned}$$

Other topics in Chapter 2 include the infinite product representations for $\sin x$ and $\cos x$ and related infinite products, as well as the approximation of roots of polynomials by convergents of continued fractions.

At the beginning of Chapter 3 of the first Notebook, we find the evaluation

$$\begin{aligned}
\int x^{n-1}e^x\,dx &= e^x\left\{\int x^{n-1}dx - \int\int x^{n-1}(dx)^2 + \int\int\int x^{n-1}(dx)^3 - \&\text{c}\right\} \\
&= e^x\left\{\frac{x^n}{n} - \frac{x^{n+1}}{n(n+1)} + \frac{x^{n+2}}{n(n+1)(n+2)} - \&\text{c}\right\}.
\end{aligned}$$

The above is merely a rather unorthodox way of writing the successive integrations by parts for $x^{n-1}e^x$. The main content of Chapter 3, however, is concerned with Taylor series and various other types of expansions of functions, many related to the exponential function.

Ramanujan begins Chapter 4 of the second Notebook (Chapter 5 of the first) with a study of the functions $F_n(x)$ defined recursively by

$$F_1(x) = e^x - 1, \quad F_{n+1}(x) = e^{F_n(x)} - 1, \qquad n \geqslant 1.$$

There follow several beautiful formulas on integral transforms of which

$$\int_0^\infty x^{n-1}\{\varphi(0) - x\varphi(1) + x^2\varphi(2) - \&\text{c}\}\,dx = \frac{\pi\varphi(-n)}{\sin(\pi n)}$$

is a striking example (vol. II, p. 45). This formula as well as other work of Ramanujan on Fourier transforms is discussed by Hardy in [19]. Parseval's theorem for integrals and the inversion formulae for Fourier cosine transforms are given (vol. II, p. 46). Several Maclaurin series in $1/x$ are "evaluated" at $x=0$. Thus, for example, Ramanujan states that (vol. II, p. 44) "When $x=0$

$$\frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5} - \&\text{c} = \frac{\pi}{2} \tag{3}$$

which is same as saying $\tan^{-1}\infty = \pi/2$." Of course, the series on the left side of (3) is simply the Maclaurin series of $\tan^{-1}(1/x)$.

Chapter 5 of the second Notebook opens with some formulae that resemble the Euler-Maclaurin formula. The chapter is primarily devoted to the Bernoulli numbers B_n and various series expansions involving the Bernoulli numbers. Several properties of Bernoulli numbers are listed. On page 57, we find Euler's formula for $S_{2n} = \zeta(2n)$ and similar formulae. The values $S_2, \ldots, S_{10}$ are calculated to 10 decimal places. However, Legendre had long ago calculated $S_2, \ldots, S_{35}$ to 16 decimal places, and Stieltjes had calculated $S_2, \ldots, S_{70}$ to 32 decimal places [47]. Several striking series are evaluated in terms of S_n (pp. 59–61). For example, for $n \geqslant 2$,

$$\frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{8^n} + \frac{1}{11^n} + \frac{1}{12^n} + \&\text{c} = \frac{S_n^2 - S_{2n}}{2S_n},$$

where the sum is over those natural numbers containing an odd number of prime factors in their factorizations. Most of these sums are contained in [31], [40, pp. 20–21]. Bernoulli numbers of fractional index are defined by generalizing Euler's formula for S_{2n}. In other words, Ramanujan defines B_n, where $n>1$ is real, by $B_n = 2\underline{|n}\, S_n/(2\pi)^n$. The Euler numbers E_{2n} may be defined by

$$\sec x = \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{(2n)!} x^{2n} \qquad (|x| < \pi/2),$$

and Ramanujan develops a theory of Euler numbers analogous to his theory of Bernoulli numbers.

Chapter 6 of the second Notebook (Chapter 8 of the first) is, on a first examination, perhaps the most mysterious chapter in the Notebooks. Here, Ramanujan expounds his strange theory of divergent series. This theory depends upon a version of the Euler-Maclaurin summation formula wherein the number of approximating terms involving Bernoulli numbers is taken to be infinite. In Ramanujan's theory, a "constant" is associated to certain divergent series. He concludes his discussion of the "constant" with the observation (vol. I, p. 79), "The constant of a series has some mysterious connection with the given infinite series and it is like the centre of gravity of a body. Mysterious because we may substitute it for the divergent infinite series." He then gives some examples. The "constant" for $1+1+1+1$ &c is said to be $-1/2$. He later defines a series to be "corrected" when its "constant" is subtracted from it. Ramanujan's theory of divergent series has been placed on a firm foundation by Hardy [22, Section 13.15]. However, he does not mention the connection between his theory and Ramanujan's discourse in the Notebooks.

Chapter 7 in the second Notebook is a continuation of Chapter 5 of the same Notebook. There are several theorems concerning $\varphi_r(x) = 1^r + 2^r + \cdots + x^r$. Bernoulli numbers of negative index are defined by means of the functional equation of the Riemann zeta-function $\zeta(s)$ for real values of s, given on page 78 of the second volume. Similarly, the functional equation of $\sum_{k=1}^{\infty}(-1)^k(2k+1)^{-s}$, $s>0$, is given on page 86 in the notation of Euler numbers, and hence this equation is used to define Euler numbers of negative index. The chapter concludes with several standard properties of the gamma function, including Stirling's formula.

Chapter 10 of the first Notebook and Chapter 8 of the second do not correspond very well. In particular, Chapter 10 contains material not found in the revised edition. Euler's constant, usually written in decimal notation, is found in several places throughout the Notebooks, although it is never given any name. In Chapter 10 of the first Notebook, we find its decimal expansion .57721566490153286060..., which is correct to the number of decimal places given. Several series are evaluated in terms of logarithms. One of the simpler examples is (vol. II, p. 94)

$$\frac{\sqrt{3}-1}{1} - \frac{(\sqrt{3}-1)^4}{4} + \frac{(\sqrt{3}-1)^7}{7} - \&\text{c} = \frac{\pi}{4\sqrt{3}} + \frac{1}{3}\log\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right).$$

Ramanujan also evaluates some integrals including

$$A_n = \int_0^x \frac{dx}{1+x^n}$$

for $n = 1-6$, 8, and 10. The formula for A_3 (vol. II, p. 94) can be used to derive the series evaluation given immediately above. The formula

$$A_5 = \frac{1}{20}\log\frac{(1+x)^5}{1+x^5} + \frac{1}{4\sqrt{5}}\log\frac{1+x\dfrac{\sqrt{5}-1}{2}+x^2}{1-x\dfrac{\sqrt{5}-1}{2}+x^2}$$

$$+\frac{1}{10}\sqrt{10-2\sqrt{5}}\,\tan^{-1}\frac{x\sqrt{10-2\sqrt{5}}}{4-x(\sqrt{5}+1)} + \frac{\sqrt{10+2\sqrt{5}}}{10}\tan^{-1}\frac{x\sqrt{10+2\sqrt{5}}}{4+x(\sqrt{5}-1)}$$

should convey to the reader an appreciation of Ramanujan's stamina for computation. On page 97 of the second volume, we find the highly unusual product evaluation

$$\frac{\left(\frac{\sqrt[1]{1}}{\sqrt{2}}\cdot\frac{\sqrt[3]{3}}{\sqrt[4]{4}}\cdot\frac{\sqrt[5]{5}}{\sqrt[6]{6}}\cdot\frac{\sqrt[7]{7}}{\sqrt[8]{8}}\,\&\text{c}\right)^{1/\log 2}}{\left(\frac{\sqrt[1]{1}}{\sqrt[3]{3}}\cdot\frac{\sqrt[5]{5}}{\sqrt[7]{7}}\cdot\frac{\sqrt[9]{9}}{\sqrt[11]{11}}\cdot\frac{\sqrt[13]{13}}{\sqrt[15]{15}}\,\&\text{c}\right)^{4/\pi}}=\frac{\sqrt{2}}{\pi}\left(\underline{|-1/4}\right)^4.$$

Ramanujan also remarks (vol. II, p. 95) that "$\underline{|x}$ is minimum when $x=6/13$ very nearly." Since $6/13=.46153\ldots$, this is very close to the actual minimum $.46163\ldots$.

The next chapter is largely devoted to trigonometric series and contains some of the most interesting formulas found in the Notebooks. On page 145 of volume 1, we find that

$$\cos 2x-\left(1+\frac{1}{3}\right)\cos 4x+\left(1+\frac{1}{3}+\frac{1}{5}\right)\cos 6x-\&\text{c}$$
$$=\frac{\pi}{4}(\cos x-\cos 3x+\cos 5x-\&\text{c}),$$

but neither side converges. Some interesting expansions in powers of trigonometric functions are found. This next intriguing relation (vol. II, p. 112)

$$\sum_{n=1}^{\infty}\frac{1}{n^2}(x^n\cos n\theta+y^n\cos n\varphi)$$
$$=\frac{1}{8}\log(1-2x\cos\theta+x^2)\log(1-2y\cos\varphi+y^2)-\frac{1}{2}\tan^{-1}\frac{x\sin\theta}{1-x\cos\theta}\tan^{-1}\frac{y\sin\varphi}{1-y\cos\varphi}$$

is supposedly valid under the assumptions $x\cos\theta+y\cos\varphi=xy\cos(\theta+\varphi)$ and $x\sin\theta+y\sin\varphi=xy\sin(\theta+\varphi)$. Many relations involving the dilogarithm $\sum_{n=1}^{\infty}x^n/n^2$, $|x|\leqslant 1$, are listed with no hints of verification. Proofs of most of these relations and many other properties of the dilogarithm may be found in Lewin's treatise **[24]**.

Chapters 10 and 11 of the second Notebook, or Chapters 12 and 13 of the first, develop the theory of hypergeometric series. These chapters have been thoroughly examined by Hardy **[17]**. Ramanujan independently derived practically all of the major classical theorems in the subject, including theorems of Gauss, Kummer, Dougall, Whipple, Dixon, Saalschütz, Schäfli, Clausen, and Thomae. In most instances, Ramanujan discovered the theorems later than those whose theorems bear their names. For proofs of most of the theorems in these two chapters, see Hardy's paper **[17]** or Bailey's tract **[4]**. Hardy **[18]** has also pointed out that many series summed by Ramanujan can be evaluated by using theorems about hypergeometric series.

Continued fraction expansions are discussed in the next chapter. We cite just two formulae. The first (vol. II, p. 147)

$$\sqrt{\frac{2x}{\pi}}-\frac{x}{1+}\,\frac{2x}{2+}\,\frac{3x}{3+}\,\frac{4x}{4+}\,\&\text{c}=\frac{2}{3\pi}\quad\text{when}\quad x=\infty$$

is an example to illustrate a theorem which "is only approximately true." The second (vol. II, p. 149)

$$\frac{\underline{\left|\frac{x+m+n-1}{2}\right.}\;\underline{\left|\frac{x-m-n-1}{2}\right.}-\underline{\left|\frac{x+m-n-1}{2}\right.}\;\underline{\left|\frac{x-m+n-1}{2}\right.}}{\underline{\left|\frac{x+m+n-1}{2}\right.}\;\underline{\left|\frac{x-m-n-1}{2}\right.}+\underline{\left|\frac{x+m-n-1}{2}\right.}\;\underline{\left|\frac{x-m+n-1}{2}\right.}}$$

$$=\frac{mn}{x+}\,\frac{(m^2-1^2)(n^2-1)}{3x+}\,\frac{(m^2-2^2)(n^2-2^2)}{5x+}\,\frac{(m^2-3^2)(n^2-3^2)}{7x+\&\text{c}}$$

is one of many similar continued fraction expansions given by Ramanujan. Several of the continued fraction expansions appear to be related to hypergeometric functions. Some of Ramanujan's theorems bear a resemblance to continued fraction expansions found in the book of Khovanskii [23]. However, most of the results in this chapter apparently have not been proved in print.

Chapter 13 of the second Notebook concentrates on theorems akin to the Fourier integral theorems, on theorems similar to Parseval's theorem, and on the evaluation of definite integrals. Most of the results are not difficult to prove. As an example, if $\alpha\beta=\pi$, then (p. 160)

$$\sqrt{\alpha}\int_0^\infty \frac{e^{-x^2}}{e^{\alpha x}+e^{-\alpha x}}\,dx=\sqrt{\beta}\int_0^\infty \frac{e^{-x^2}}{e^{\beta x}+e^{-\beta x}}\,dx.$$

This identity was one of the first of several questions that Ramanujan published in the Journal of the Indian Mathematical Society [29], [40, p. 324]. See also [35], [40, pp. 53–58]. The Poisson summation formula (p. 165) and other summation formulas are found in this chapter. Classical integral representations of the Bernoulli and Euler numbers are given (p. 158).

With the exception of the chapters on hypergeometric series, the contents of Chapter 14 in the second Notebook have served as the basis of more research papers than any other topic of the Notebooks. It is quite curious, however, that many authors were unaware that their discoveries were hidden in the Notebooks. A few of the results in this chapter are found in Ramanujan's letters to Hardy [40, p. xxvi]. The chapter chiefly concerns relations between certain types of series. One of the formulas of the chapter that has been thoroughly scrutinized in recent times is Ramanujan's famous formula for $\zeta(2n+1)$. If α and β are positive numbers with $\alpha\beta=\pi^2$, and if n is any natural number, then (p. 177)

$$\alpha^{-n}\left\{\frac{1}{2}\zeta(2n+1)+\sum_{k=1}^\infty \frac{k^{-2n-1}}{e^{2\alpha k}-1}\right\}=(-\beta)^{-n}\left\{\frac{1}{2}\zeta(2n+1)+\sum_{k=1}^\infty \frac{k^{-2n-1}}{e^{2\beta k}-1}\right\}$$
$$-2^{2n}\sum_{k=0}^{n+1}(-1)^k\frac{B_{2k}}{(2k)!}\,\frac{B_{2n+2-2k}}{(2n+2-2k)!}\alpha^{n+1-k}\beta^k.$$

As is well known, the arithmetical nature of $\zeta(2n+1)$ is unknown. The above formula for $\zeta(2n+1)$ is probably the most interesting known formula for $\zeta(2n+1)$. For references to the many proofs of Ramanujan's formula, see [6] or [7] by the author. We cite two more examples from the wealth of fascinating formulae in this chapter. If $\alpha\beta=\pi$, then (p. 169)

$$\frac{\alpha}{4}\coth(n\alpha)-\frac{\beta}{4}\cot(n\beta)=\frac{n}{2}+\alpha\sum_{k=1}^\infty\frac{\sinh(2kn\alpha)}{e^{2k\alpha^2}-1}+\beta\sum_{k=1}^\infty\frac{\sin(2kn\beta)}{e^{2k\beta^2}-1}.$$

(No conditions on n are given, but the formula only holds if $0<\beta n<\pi$.) With no conditions listed for x and y (p. 175),

$$\frac{\pi}{4}\tan\left(\frac{\pi x}{2}\right)\tanh\left(\frac{\pi y}{2}\right)=y^2\sum_{k=1}^\infty\frac{\tanh\{(2k-1)\pi x/2y\}}{(2k-1)\{(2k-1)^2+y^2\}}+x^2\sum_{k=1}^\infty\frac{\tanh\{(2k-1)\pi y/2x\}}{(2k-1)\{(2k-1)^2-x^2\}}.$$

Series of the type

$$\sum_{k=1}^\infty\frac{\coth(k\pi)}{k^{4n-1}}\quad\text{and}\quad\sum_{k=0}^\infty\frac{(-1)^k\operatorname{sech}\{(2k+1)\pi/2\}}{(2k+1)^{4n+1}},$$

where n is a positive integer, are summed in closed form. A good reference for sums of this type is a paper of Nanjundiah [**27**]. Another interesting result is (p. 168)

$$\sum_{k=1}^{\infty} \frac{\sin^{2n+1}kx}{k} = \frac{\sqrt{\pi}}{2} \frac{\lfloor n-1/2}{\lfloor n},$$

where n is a natural number and $0 < x < \pi/(n+1)$. For proofs of several of the results of this chapter and references to other proofs, see [**7**].

Chapter 15 of the second Notebook is concerned with at least three apparently unrelated topics. There are some formulas which resemble those of the previous chapter. Ramanujan defines several concepts in the continuation of his unusual theory of series. Formulae from the theory of elliptic functions are also given.

Two related topics are developed in the following chapter. The first four pages develop the theory of the "partition" function

$$\prod(a,x) = (1+a)(1+ax)(1+ax^2)(1+ax^3)\cdots.$$

$\prod(a,x)$ also has the series representation

$$\sum_{n=0}^{\infty} \frac{a^n x^{n(n-1)/2}}{(1-x)(1-x^2)\cdots(1-x^n)}$$

and is called a basic hypergeometric function. On the chapter's opening page, we find classical results due to Cauchy, Heine and several others. Many of the theorems on $\Pi(a,x)$ can be found in Bailey's tract [**4**]. On page 196 of the second volume, there appears, in the words of Hardy [**21**, p. 222], "a remarkable formula with many parameters" involving several basic hypergeometric functions. This important formula contains the famous Jacobi triple product identity as a special case. For an especially elementary proof of Ramanujan's result and references to other proofs, see a paper of Andrews and Askey [**2**].

The second topic of Chapter 16 is a thorough development of the theory of

$$f(a,b) = 1 + (a+b) + ab(a^2+b^2) + (ab)^3(a^3+b^3) + (ab)^6(a^4+b^4) + \cdots,$$

which is actually one of the standard theta-functions. Ramanujan's work in this area of elliptic functions is well described in Chapter 12 of Hardy's book [**21**].

In Chapter 17 of the second Notebook, and in Chapter 15 of the first, Ramanujan develops the theory of

$$\exp\left[-\pi \frac{1+\left(\frac{1}{2}\right)^2(1-x)+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2(1-x)^2+\&c}{1+\left(\frac{1}{2}\right)^2 x+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x^2+\&c}\right],$$

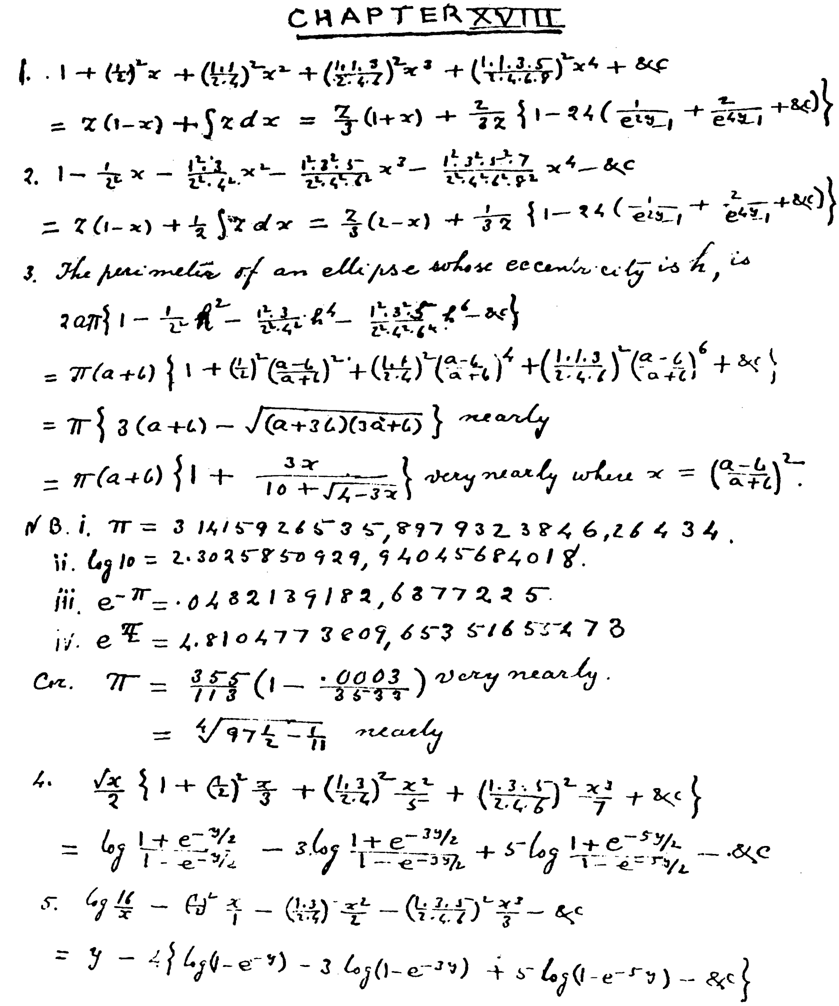
which, in contemporary notation, is equal to

$$\exp\left(-\pi \frac{F(1/2,1/2;1;1-x)}{F(1/2,1/2;1;x)}\right),$$

where $F(a,b;c;x)$ denotes the ordinary hypergeometric function. Some beautiful relations between integrals are found (vol. II, pp. 207–209). For example, if $\tan\alpha/\tan\beta = \sqrt{1+x}$, then

$$\int_0^{\alpha} \frac{d\varphi}{\sqrt{1+x\cos 2\varphi}} = \int_0^{\beta} \frac{d\varphi}{\sqrt{1-x^2\sin^4\varphi}}.$$

Most of the chapter lies in the domain of elliptic functions.



CHAPTER XVIII

1. $1 + (\frac{1}{2})^2 x + (\frac{1.1}{2.4})^2 x^2 + (\frac{1.1.3}{2.4.6})^3 x^3 + (\frac{1.1.3.5}{2.4.6.8})^2 x^4 + \&c$

$= z(1-x) + \int z\,dx = \frac{z}{3}(1+x) + \frac{2}{3z}\left\{1 - 24\left(\frac{1}{e^{2y}-1} + \frac{2}{e^{4y}-1} + \&c\right)\right\}$

2. $1 - \frac{1}{2^2}x - \frac{1^2.3}{2^2.4^2}x^2 - \frac{1^2.3^2.5}{2^2.4^2.6^2}x^3 - \frac{1^2.3^2.5^2.7}{2^2.4^2.6^2.8^2}x^4 - \&c$

$= z(1-x) + \frac{1}{2}\int z\,dx = \frac{z}{3}(2-x) + \frac{1}{3z}\left\{1 - 24\left(\frac{1}{e^{2y}-1} + \frac{2}{e^{4y}-1} + \&c\right)\right\}$

3. The perimeter of an ellipse whose eccentricity is h, is

$2a\pi\left\{1 - \frac{1}{2^2}h^2 - \frac{1^2.3}{2^2.4^2}h^4 - \frac{1^2.3^2.5}{2^2.4^2.6^2}h^6 - \&c\right\}$

$= \pi(a+b)\left\{1 + (\frac{1}{2})^2(\frac{a-b}{a+b})^2 + (\frac{1.1}{2.4})^2(\frac{a-b}{a+b})^4 + (\frac{1.1.3}{2.4.6})^2(\frac{a-b}{a+b})^6 + \&c\right\}$

$= \pi\left\{3(a+b) - \sqrt{(a+3b)(3a+b)}\right\}$ nearly

$= \pi(a+b)\left\{1 + \frac{3x}{10 + \sqrt{4-3x}}\right\}$ very nearly where $x = (\frac{a-b}{a+b})^2$.

N B. i. $\pi = 3.14159265358979323846,26434$.

ii. $\log 10 = 2.302585092994045684018$.

iii. $e^{-\pi} = .0432139182,6377225$.

iv. $e^{\pi} = 4.8104773809,6535165547\,73$

Cr. $\pi = \frac{355}{113}(1 - \frac{.0003}{3533})$ very nearly.

$= \sqrt[4]{97\frac{1}{2} - \frac{1}{11}}$ nearly

4. $\frac{\sqrt{x}}{2}\left\{1 + (\frac{1}{2})^2\frac{x}{3} + (\frac{1.3}{2.4})^2\frac{x^2}{5} + (\frac{1.3.5}{2.4.6})^2\frac{x^3}{7} + \&c\right\}$

$= \log\frac{1+e^{-y/2}}{1-e^{-y/2}} - 3\log\frac{1+e^{-3y/2}}{1-e^{-3y/2}} + 5\log\frac{1+e^{-5y/2}}{1-e^{-5y/2}} - \&c$

5. $\log\frac{16}{x} - (\frac{1}{2})^2\frac{x}{1} - (\frac{1.3}{2.4})^2\frac{x^2}{2} - (\frac{1.3.5}{2.4.6})^2\frac{x^3}{3} - \&c$

$= y - 4\left\{\log(1-e^{-y}) - 3\log(1-e^{-3y}) + 5\log(1-e^{-5y}) - \&c\right\}$

Approximations to π.

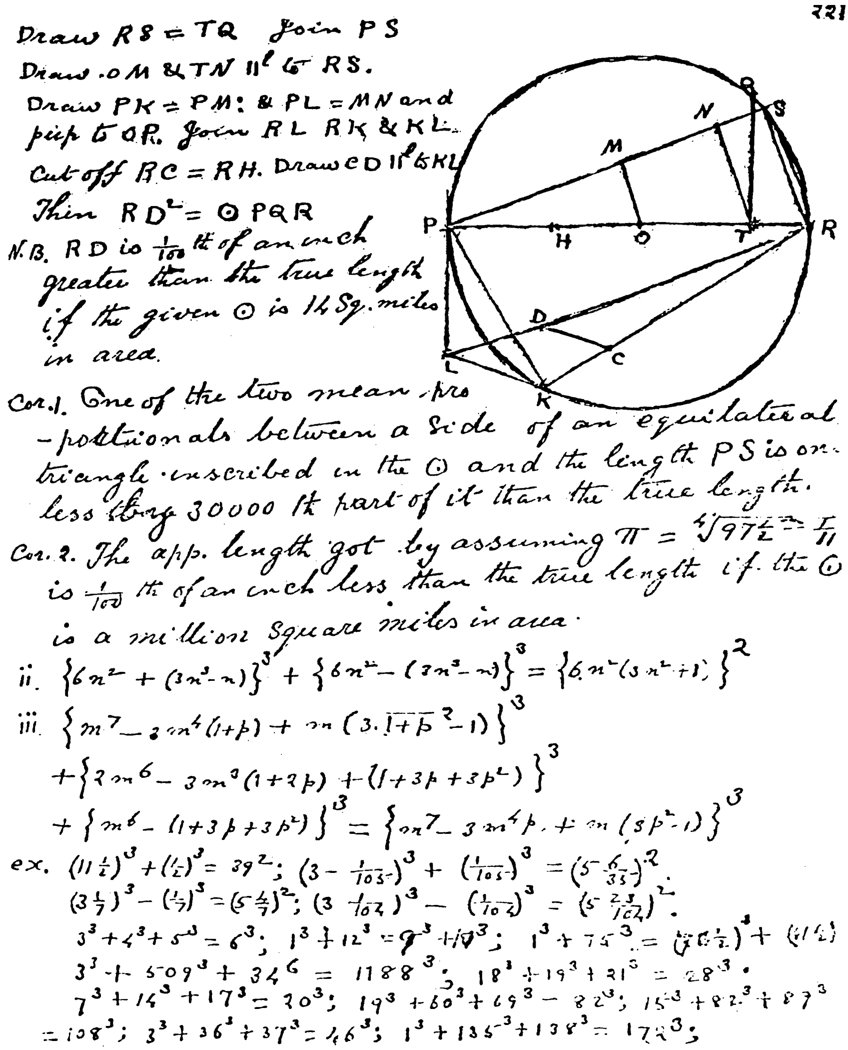
According to Ramanujan (vol. II, Chapter 18, p. 217),

$$\pi = \frac{355}{113}\left(1 - \frac{.0003}{3533}\right) \quad \text{very nearly, and} \quad \pi = \sqrt[4]{97\tfrac{1}{2} - \tfrac{1}{11}} \quad \text{nearly.}$$

In this instance, "very nearly" means that the approximation is good to 15 decimal places, and "nearly" means that the approximation is valid to 9 decimal places. These approximations to π, as well as other curious approximations, can be found in Ramanujan's paper [**34**], [**40**, pp. 23–39]. The latter approximation to π given above is used later in the chapter (p. 225) where Ramanujan indicates a method for almost squaring the circle. See also a one page paper [**33**], [**40**, p. 22] that Ramanujan wrote on squaring the circle.

Several pages of Chapter 18 are devoted to series involving trigonometric and hyperbolic trigonometric functions. A typical example is (p. 218)

$$\sum_{n=0}^{\infty} (-1)^n \frac{\cos\{(2n+1)\theta\} + 2\cos\{\frac{1}{2}(2n+1)\theta\}\cosh\{\frac{1}{2}(2n+1)\sqrt{3}\,\theta\}}{(2n+1)\cosh\{\frac{1}{2}(2n+1)\sqrt{3}\,\pi\}} = \frac{\pi}{8}.$$


221

Draw RS = TQ Join PS
Draw OM & TN ||ˡ to RS.
Draw PK = PM; & PL = MN and perp to OP. Join RL RK & KL.
Cut off RC = RH. Draw CD ||ˡ to KL
Then $RD^2 = \odot PQR$
N.B. RD is $\frac{1}{100}$ th of an inch greater than the true length if the given ⊙ is 14 Sq. miles in area.

Cor. 1. One of the two mean proportionals between a side of an equilateral triangle inscribed in the ⊙ and the length PS is only less than 30000 th part of it than the true length.

Cor. 2. The app. length got by assuming $\pi = \sqrt[4]{97\frac{1}{2} - \frac{1}{11}}$ is $\frac{1}{100}$ th of an inch less than the true length if the ⊙ is a million square miles in area.

ii. $\{6n^2 + (3n^3 - n)\}^3 + \{6n^2 - (3n^3 - n)\}^3 = \{6n^2(3n^2+1)\}^2$

iii. $\{m^7 - 3m^4(1+p) + m(3\overline{1+p}^2 - 1)\}^3$
$+\{2m^6 - 3m^3(1+2p) + (1+3p+3p^2)\}^3$
$+\{m^6 - (1+3p+3p^2)\}^3 = \{m^7 - 3m^4p + m(3p^2-1)\}^3$

ex. $(11\frac{1}{2})^3 + (\frac{1}{2})^3 = 39^2$; $(3 - \frac{1}{105})^3 + (\frac{1}{105})^3 = (5 - \frac{6}{35})^2$.
$(3\frac{1}{7})^3 - (\frac{1}{7})^3 = (5\frac{4}{7})^2$; $(3 - \frac{1}{102})^3 - (\frac{1}{102})^3 = (5 - \frac{23}{102})^2$.
$3^3 + 4^3 + 5^3 = 6^3$; $1^3 + 12^3 = 9^3 + 10^3$; $1^3 + 75^3 = (56\frac{1}{2})^3 + (61\frac{1}{2})^3$
$3^3 + 509^3 + 34^6 = 1188^3$; $18^3 + 19^3 + 21^3 = 28^3$.
$7^3 + 14^3 + 17^3 = 20^3$; $19^3 + 60^3 + 69^3 = 82^3$; $15^3 + 82^3 + 87^3 = 108^3$; $3^3 + 36^3 + 37^3 = 46^3$; $1^3 + 135^3 + 138^3 = 172^3$;

The source of the taxicab anecdote.

A simpler example (p. 219)

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^7 \cosh\left\{\frac{1}{2}(2n+1)\sqrt{3}\,\pi\right\}} = \frac{\pi^7}{23{,}040}$$

was established by Watson [**48**].

Most readers are undoubtedly familiar with Hardy's famous story about the taxicab number [**40**, p. xxxv], [**21**, p. 12]. On the way to visiting Ramanujan, when he was seriously ill, Hardy rode in a taxicab numbered 1729. Hardy expressed his feelings to Ramanujan that the number is rather a dull one. But Ramanujan exclaimed that 1729 is, indeed, very interesting, for 1729 is the first natural number that can be written as the sum of two cubes in two distinct ways. On the bottom of page 225 in the second Notebook, we find these representations

$$1729 = 1^3 + 12^3 = 9^3 + 10^3.$$

Ramanujan also gives other examples of this sort, like

$$19^3 + 60^3 + 69^3 = 82^3 \quad \text{and} \quad 133^3 + 174^3 = 45^3 + 196^3.$$

Chapters 19–21 in the second volume concern elliptic functions and are a continuation of the theory developed in Chapters 16 and 17. Several pages consist solely of very complicated expressions involving radicals. These are computations involving singular moduli. The material is related to Ramanujan's work in [**34**], [**40**, pp. 23–39]. Watson has made a thorough study of that material in Chapters 16–21 which is connected with elliptic functions. See the several papers of Watson listed in the bibliography of [**21**], as well as Chapter 12 there.

The bulk of the miscellaneous material after Chapter 16 in the first Notebook is related to elliptic functions. On page 274 of the second volume, the germ for Ramanujan's paper [**36**], [**40**, pp. 59–67] is found in that

$$\int_0^\infty \frac{\cos nx}{e^{2\pi\sqrt{x}}-1}\,dx$$

is evaluated for certain values of n. See also [**32**], [**40**, p. 327] and a paper of Watson [**50**]. The miscellaneous notes also contain additional integral theorems of the Fourier type, some work on functional equations, several additional continued fraction expansions, definitions of legitimate and illegitimate convergent and divergent series (p. 348), and more theorems related to Chapters 14 and 16–21.

Some of Ramanujan's work on number theory is found in the miscellaneous material. Ramanujan, no doubt, did not have proofs of many of his statements in the theory of numbers. In fact, some of his claims are false. But his insights and intuition are remarkable. On page 307 (volume II), Ramanujan gives a form of the prime number theorem by stating that "The no. of prime nos. between A and B

$$=\int_A^B \frac{dx}{\log x} \qquad \text{nearly."}$$

(Other estimates for the number of primes can be found on pages 317–318.) He also says that the number of integers between A and B that can be written as a sum of two squares is

$$C\int_A^B \frac{dx}{\sqrt{\log x}} \qquad \text{nearly,}$$

where $C=.764\ldots$. This statement is asymptotically true, and a proof of this was first given by Landau [**21**, pp. 62–63]. The result was also communicated to Hardy in one of Ramanujan's letters [**40**, p. xxiv]. Hardy discusses this statement at length [**21**, pp. 8–9, 19, 62–63], and apparently there is some controversy about how good of an approximation Ramanujan thought that he had. For a full discussion of this and references to recent work, see a paper of Shanks [**45**]. Ramanujan gives representations of primes as sums or differences of squares, but most of these results were discovered long ago by Fermat.

The third Notebook is somewhat of a continuation of the heterogeneous material at the end of the second Notebook. More material on sums of squares and divisors of integers is found. On page 371, Ramanujan gives some long intervals of composite numbers, for example, 370,261 to 370,373 and 2,010,733 to 2,010,881. As usual, let $\pi(x)$ denote the number of primes not exceeding x. Ramanujan calculated $\pi(x)$ for several values of x. For example, Ramanujan claims that $\pi(10^8)=5{,}761{,}460$; the correct value is 5,761,455. The seeds for Ramanujan's paper on highly composite numbers [**37**], [**40**, pp. 78–128] are found on page 372.

On page 374, Ramanujan announces some expressions for certain values of the exponential function. For example, he states that

$$e^{\frac{\pi}{4}\sqrt{78}}=4\sqrt{3}\,(75+52\sqrt{2}\,) \tag{4}$$

and

$$e^{\frac{\pi}{4}\sqrt{130}}=12(323+40\sqrt{65}\,). \tag{5}$$

These and similar findings are found in Ramanujan's paper [**34**], [**40**, pp. 23–39], where he writes that these are "approximate formulae". Samuel Wagstaff has kindly computed the numbers in (4) and (5). The left and right hand sides of (4) and (5), respectively, are

$$1029.1091087457\ldots, \qquad 1029.1091087695\ldots$$

and

$$7745.88371918324\ldots, \qquad 7745.88371918330\ldots.$$

On page 375, we find that

$$e^{\pi\sqrt{58}} = 24591257751.99999982\ldots$$

is "almost" an integer. Examples like those in this paragraph can be explained by means of the theory of complex multiplication and elliptic functions. Moreover, instances of this sort are found in the works of Hermite and Kronecker [**46**, p. 357] several years earlier. A particular example of this type of approximation recently served as the basis of an "April Fool's" joke by Martin Gardner [**14**, p. 126] in *Scientific American.*

Examples of equalities between fourth and higher powers of integers are found on pages 384–386 with some general rules on how to construct such examples.

Half of page 390 and pages 391 and 392 are written upside down. Page 391 is somewhat enigmatic. As mentioned earlier, Ramanujan never indicated any knowledge of complex analysis. Phrases such as "It can be shewn, by the theory of residues, that..." [**38**], [**40**, p. 129] are thought to have been supplied by Hardy. On page 391, however, Ramanujan states several Mellin transforms, and the words "contour integration" appear. Furthermore, Ramanujan also uses the notation $\Gamma(n)$, where elsewhere in the Notebooks, Ramanujan would have used $\underline{|n-1}$ instead. Probably, this single page dates from Ramanujan's years at Cambridge.

Epilogue

In discussions about Ramanujan, the question "How really great a mathematician was he?" inevitably arises. In the domains of infinite series, elliptic functions, and continued fractions, very few in the history of mathematics have been his equal. Our judgement, however, is somewhat clouded by the singular nature of his "proofs". Hardy [**21**, p. 7] deemed that "It (Ramanujan's mathematics) has not the simplicity and the inevitableness of the very greatest work; it would be greater if it were less strange." Much of Ramanujan's work which did not seem inevitable at the time it was discovered has now become more inevitable as we see how it fits into the rest of mathematics. Thus, perhaps the hour of final judgement is not yet at hand. But of Ramanujan's love and devotion for mathematics, there can be no doubt. And perhaps he would have said with Shelley, in the latter's Hymn to Intellectual Beauty,

I vowed that I would dedicate my powers
To thee and thine; have I not kept the vow?

This paper has benefited from the comments of several mathematicians. In particular, the author wishes to thank George Andrews, Richard Askey, Shashi Gadia, Emil Grosswald, and Kenneth Stolarsky.

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Unstable Polyhedral Structures

Unlike convex polyhedra, some non-convex forms may shake, snap or flex.

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It was shown by Cauchy [1] and Dehn [2] that a convex polyhedron made of rigid plates which are hinged at their edges is a rigid structure. However, if the structure is not convex, but still simply connected, there are several possibilities. It may be any of the following cases:

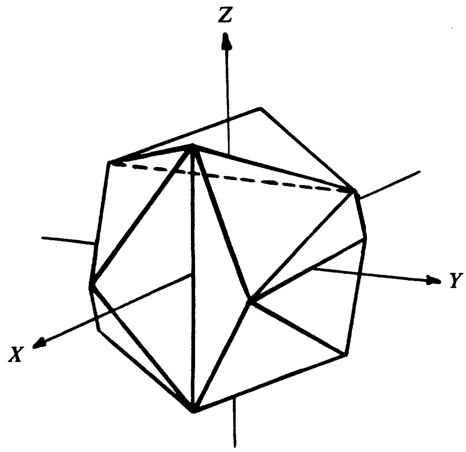
(a) rigid,

(b) infinitesimally movable (shaky),

(c) two or more stable forms (multi-stable),

(d) a continuously movable linkage.

Shaky polyhedra.



Orthogonal icosahedron (shaky polyhedron)

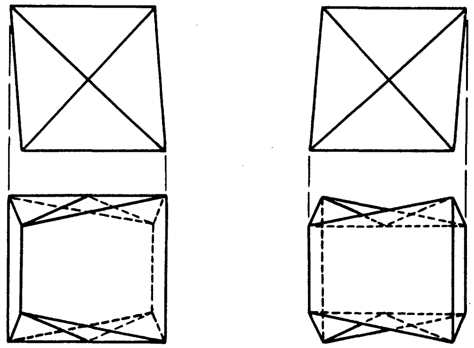
FIGURE 1.

The regular icosahedron of twenty triangular faces is convex and rigid. If six pairs of faces with an edge in common are replaced by other pairs of isosceles faces with their edges in common at right angles to the original common edge to make a non-convex icosahedron, we may obtain the orthogonal icosahedron of Jessen [3], shown in FIGURE 1. Each dihedral angle is $\pi/2$ or $3\pi/2$. This polyhedron can be deformed infinitesimally, making the dihedral angles at all the long edges infinitesimally greater or less, without strain on the faces. Structures of this type are called **shaky** structures. The tensegrity icosa of R. Buckminster Fuller [4] is made by replacing the long edges by rigid columns in compression, while the short edges are replaced by wires in tension. A shaky octahedron was described by Blaschke [5] and is pictured in a paper by Gluck [6]. Other shaky structures will be described in later sections.

Multi-stable structures

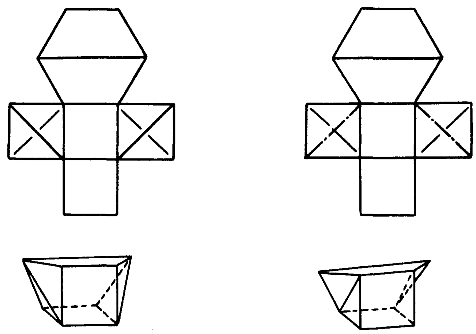
A polyhedron can be made by folding its development into a closed surface. Sometimes this can be done in more than one way. If two of these forms are nearly alike, it may be possible to change from one form to the other with only a slight transient non-destructive elastic stretch of the material. The simplest case is made by capping each end of a rhombic prism by shallow regular non-planar quadrilaterals. Each quadrilateral is bridged by four congruent isosceles triangles. The surfaces that they make may be convex or concave, as shown in FIGURE 2; either form can be buckled into the other. Another example, described by Efimow [7], is shown in FIGURE 3. (This book by Efimow has an excellent bibliography of works of many investigators including Hilbert, Cohn-Vossen and Weyl.)

Another type of example is a non-convex triangular antiprism described by Wunderlich [8]. The chain of six lateral plates, shown in FIGURE 4, can be closed to make a buckled ribbon surface in



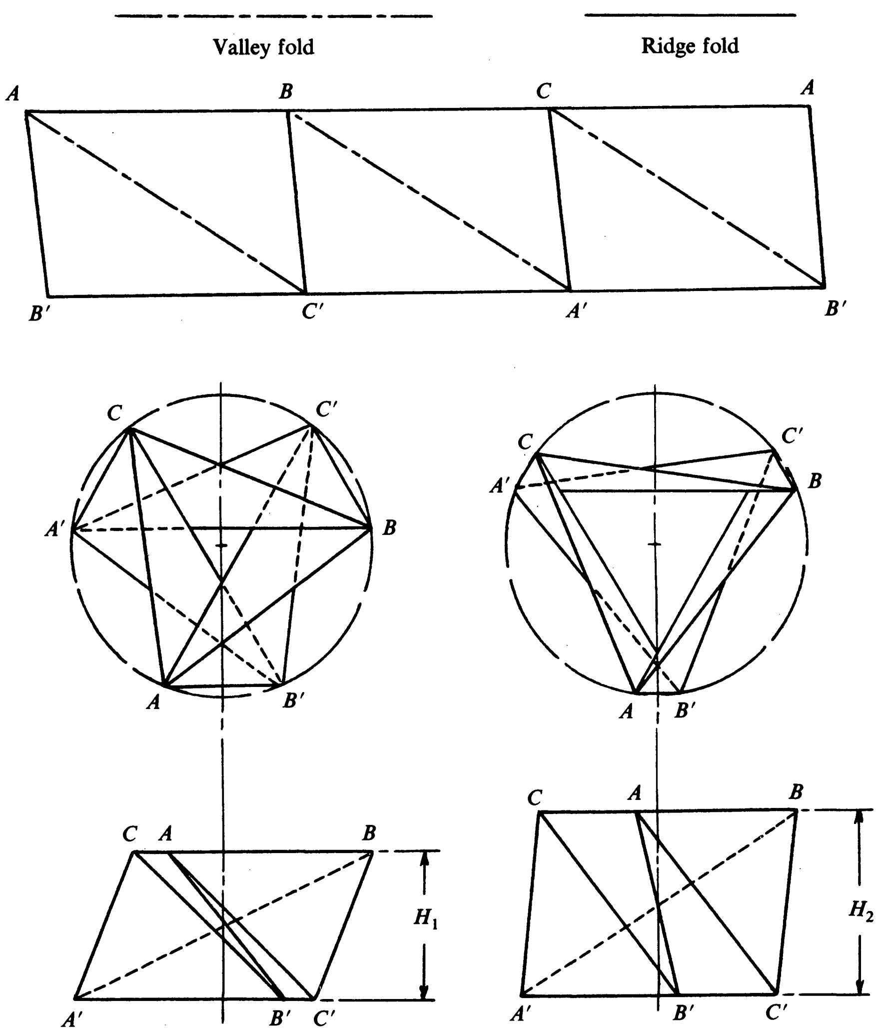
Convex and concave states of prism with skew quadrilateral bases.

FIGURE 2.



Convex and concave states of polyhedron made from same development (Efimow)

FIGURE 3.

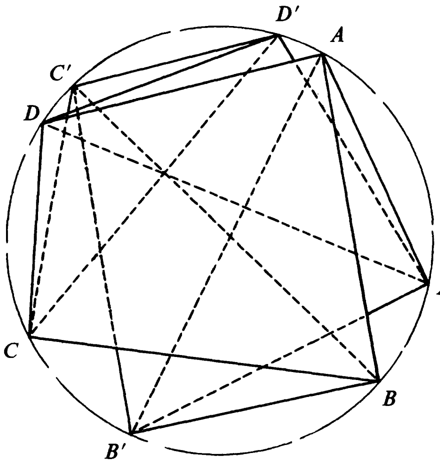


Chain of six congruent triangles and two stable states of Wunderlich triangular antiprism.

FIGURE 4.

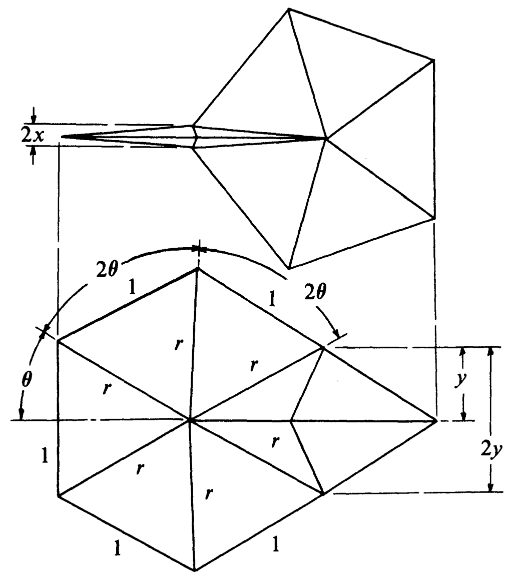
either of two ways. By adding two triangles, a complete closed octahedron is produced. Either form can be made into the other while undergoing a slight temporary elastic strain. This structure can be generalized to other antiprisms, both regular and irregular, provided the bases are congruent cyclic polygons. The simplest cases to design and construct are those in which all the faces can be collapsed into a plane. Then the true shapes of all the faces are shown, as in FIGURE 5.

In each of these examples, there is a special case in which two of the stable positions coincide. In this case, the structure becomes a shaky structure.



Collapsed state of bistable quadrilateral antiprism.

FIGURE 5.



Siamese dipyramid of 20 congruent triangles.

FIGURE 6.

The Siamese dipyramids

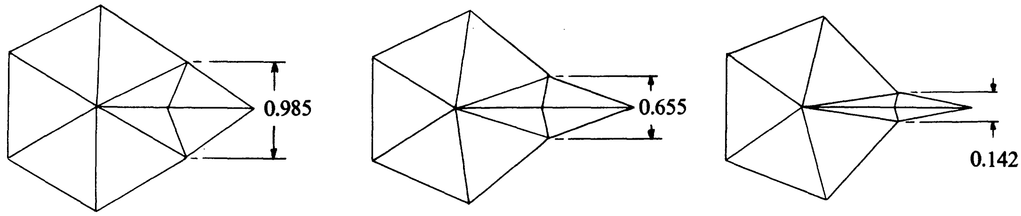
Some years ago, in his search for simply connected closed structures which have multiple stable positions, the author devised the following Siamese dipyramids. These are made of two regular dipyramids which are combined to form a closed polyhedron, as shown in FIGURE 6. If one of the dipyramids is compressed, then the other dipyramid increases in height. Consider the case of a Siamese dipyramid made of twenty equilateral triangles of unit edge. If the height of one dipyramid is $2x$, the height of the other dipyramid is $2y$, the radii of one dipyramid are r, and the angles are θ and 2θ as marked, then the variables r, θ, x, y are connected by the relations

$$\sin\theta = 1/2r, \qquad x^2 = 1 - r^2,$$
$$y = r\sin 5\theta = r(5\sin\theta - 20\sin^3\theta + 16\sin^5\theta)$$
$$= r\sin\theta(5 - 20\sin^2\theta + 16\sin^4\theta) = (5 - 5/r^2 + 1/r^4)/2.$$

Hence, $2yr^4 = 5r^4 - 5r^2 + 1$, or $2y(1-x^2)^2 = 5x^4 - 5x^2 + 1$. If $2x = 2y$, we obtain the solutions $2x = 2y \approx 0.655$. Other solutions of the equation are $2x \approx 0.142$ and $2y \approx 0.985$, or $2x \approx 0.985$ and $2y \approx 0.142$. Hence, there are three stable configurations that the polyhedron can assume.

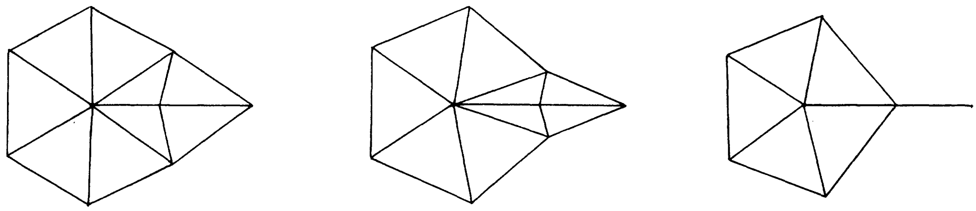
If the equilateral triangles are replaced by isosceles triangles whose apex angles are approximately 59°, instead of 60°, then either dipyramid can be completely collapsed, as shown in FIGURE 7. If the apex angles are approximately 62°, then the three stable positions coincide to make it another shaky icosahedron for which $2x = 2y$.

The foregoing examples are only special cases of a family of dipyramids of $4n$ faces. For $n = 3$, we obtain a shaky dodecahedron of twelve isosceles triangles of apex angle approximately 107°36′.



Three stable states of Siamese dipyramid of 20 congruent equilateral triangles.

FIGURE 7.

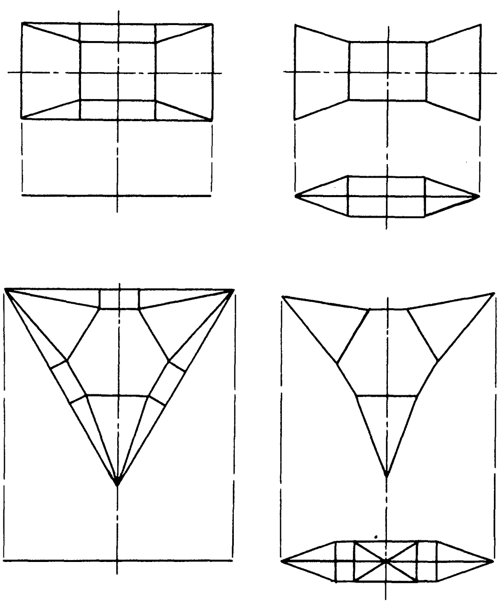


Three stable states of Siamese dipyramid of 20 congruent isosceles triangles of apex angles approximately 59°.

FIGURE 8.

Buckled surfaces

Many examples of polyhedra with several states of stability can be obtained by buckling polyhedral surfaces. For example, two congruent rectangles, that are joined along all four edges to form a closed envelope, can be buckled into a non-convex polyhedron of twelve faces, as shown in FIGURE 9. A triangular envelope can be buckled into a triangular prism to which three rectangular pyramids are joined. By successive buckling, a large number of stable states can be produced.

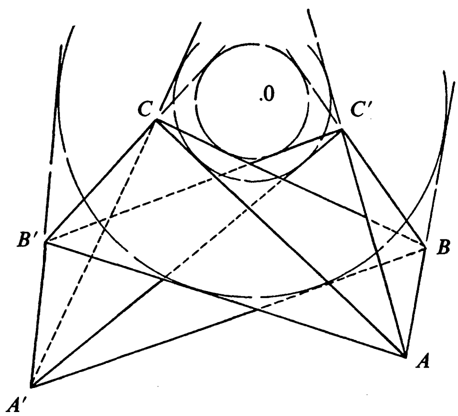


Buckled surfaces made from rectangular and triangular envelopes.

FIGURE 9.

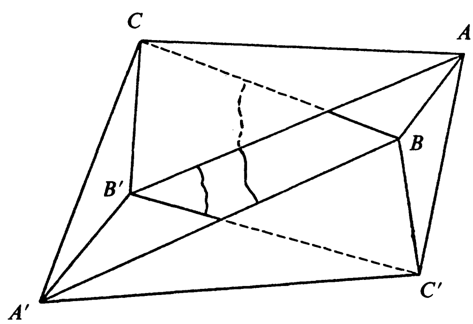
The Bricard deformable octahedron

One of the most remarkable plate linkages was discovered by Bricard [**9**, **10**]. A special form of this linkage is doubly collapsible, that is, all the faces of the linkage can be collapsed into a plane in two distinct ways. This form of the linkage can be constructed as follows. Construct two concentric circles of arbitrary radii, as shown in FIGURE 10a. Choose two arbitrary points A and A' outside of the larger circle. Construct the tangents from A and A' to the circles and determine their intersections B, B', C and C'. The lines BC, $B'C'$, $B'C$ and BC' will be tangent to a third concentric circle. Then the six triangles ABC', ABC, $AB'C$, $A'B'C$, $A'B'C'$, $A'BC'$ taken in that cyclic order, hinged at the common edges, constitute a deformable six-plate linkage. The construction of a workable cardboard model from the net of FIGURE 11 is not difficult, and it is well worth the effort. The other collapsed form taken by this linkage is shown in FIGURE 10b.



Bricard deformable octahedron (collapsed)

FIGURE 10a.



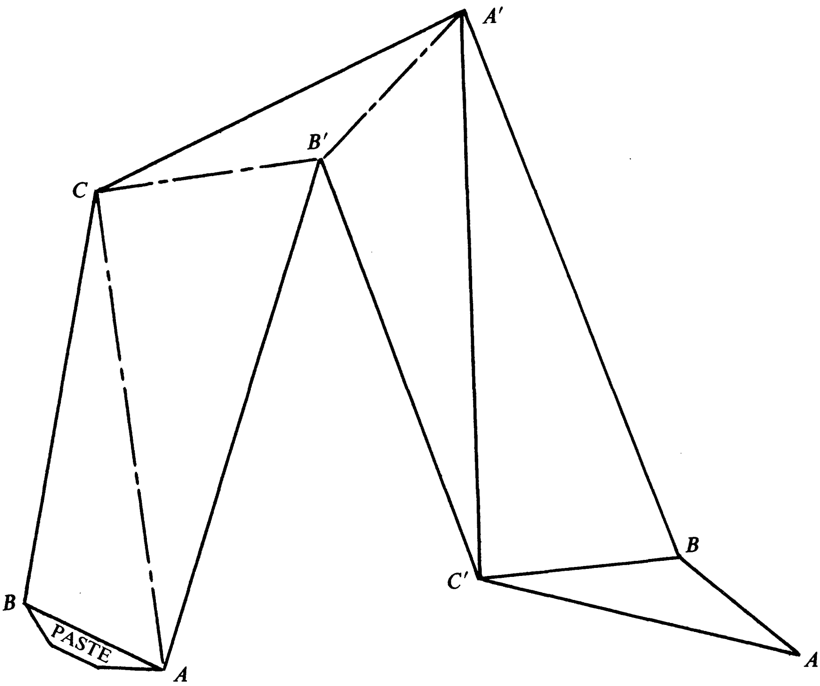
Bricard deformable octahedron (collapsed)

FIGURE 10b.

The free edges of the foregoing six-plate linkage are the edges of two triangles $AB'C'$ and $A'BC$. If these triangles are added as hinged plates to the linkage, a complete closed octahedron is formed. This octahedron is deformable in the same manner that the six-plate linkage is deformable; the added triangles impose no additional restraints. However, this octahedron is not simply connected, since the faces pass through each other. Therefore, a complete working model of material plates is not possible. Part of the missing plates $AB'C'$ and $A'BC$ can be added, as shown in FIGURE 10b, and still permit motion of the linkage.

The closed flexible polyhedron

Since the time of Cauchy, mathematicians have sought a simply connected closed polyhedral structure which is continuously movable as a linkage. Recently Robert Connelly [**13**] has produced such a linkage by an ingenious modification of the Bricard octahedron. The plates $A'B'C$ and ABC' are retained unchanged. However, on each of the other six plates, a rigid trihedral structure has been built externally, and the original six plates have been eliminated. The resulting linkage has no interference, but it has a very limited motion. The relative motions of the eight links in this linkage are the same as in the original Bricard linkage. By a further modification of two hinges, the mobility has been greatly increased. This modification consists of cutting a notch in a hinge, and then closing the opening by the addition of six small plates. This "crinkle mechanism" is described in detail in a forthcoming paper [**14**].



Net for Bricard deformable octahedron.

FIGURE 11.

Extensions and open questions

Besides the Bricard octahedron, there are other non-simple polyhedral structures that are deformable. There are many three-dimensional linkage mechanisms in which each link can be replaced by a tetrahedron or a prism. Then, the linkage becomes a deformable plate linkage. Examples are described and pictured in papers by the author [**11**, **12**]. The possibilities of such three-dimensional linkage mechanisms have not been exhausted. They are being actively investigated by mechanical engineers.

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Polynomial Roots and Matrices

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Let $p(x)$ be a monic polynomial with coefficients in a field F. Suppose that $h(x)$ is another polynomial whose roots can be expressed in terms of the roots of $p(x)$. A large part of the theory of equations is devoted to methods for computing the coefficients of $h(x)$ in terms of the coefficients of $p(x)$. The problem is not a trivial one because in general it is impossible to express the roots of a polynomial of degree greater than four in terms of its coefficients.

The usual way to solve this problem is to use the fundamental theorem of symmetric polynomials [4, p. 353]. For example, suppose that $p(x)=x^2+p_1x+p_2$ has roots λ_1, λ_2 and that we want to compute the coefficients of a polynomial $h(x)$ having roots λ_1^2, λ_2^2. It follows from the equation $x^2+p_1x+p_2=(x-\lambda_1)(x-\lambda_2)$ that $p_1=-(\lambda_1+\lambda_2)$ and $p_2=\lambda_1\lambda_2$. Thus

$$\begin{aligned} h(x) &= (x-\lambda_1^2)(x-\lambda_2^2) \\ &= x^2-(\lambda_1^2+\lambda_2^2)x+\lambda_1^2\lambda_2^2 \\ &= x^2+(2\lambda_1\lambda_2-(\lambda_1+\lambda_2)^2)x+(\lambda_1\lambda_2)^2 \\ &= x^2+(2p_2-p_1^2)x+p_2^2. \end{aligned}$$

The roots of $h(x)$ are the squares of the roots of $p(x)$, and the coefficients of $h(x)$ are polynomials in the coefficients of $p(x)$.

There is another way to approach this problem. The matrix

$$P=\begin{bmatrix} 0 & 1 \\ -p_2 & -p_1 \end{bmatrix}$$

has characteristic polynomial $p(x)$ since $\det(xI-P)=x^2+p_1x+p_2$. Moreover, the characteristic polynomial of P^2 has roots λ_1^2, λ_2^2. To find its coefficients, we compute

$$P^2=\begin{bmatrix} -p_2 & -p_1 \\ p_1p_2 & p_1^2-p_2 \end{bmatrix}$$

and

$$\det(xI-P^2)=x^2+(2p_2-p_1^2)x+p_2^2.$$

The subject of this article is the development of this matrix theoretic method. It yields an explicit formula for the polynomial $h(x)$ whereas in general the symmetric polynomial approach requires the use of an algorithm. Some of the methods presented here may be found in [3, p. 24].

The general problem is to determine for a given set of polynomials the coefficients of another polynomial whose roots depend on the roots of the given polynomials. We limit ourselves to the case in which the given set contains just two polynomials, $p(x)=x^n+p_1x^{n-1}+\cdots+p_n$ and $q(x)=x^m+q_1x^{m-1}+\cdots+q_m$, having roots $\lambda_1,\ldots,\lambda_n$ and $\mu_1,\ldots,\mu_m$, respectively. We will first show how to find the coefficients of the polynomials whose roots are $f(\lambda_1),\ldots,f(\lambda_n)$, where $f(x)$ is a polynomial. We shall deal later with finding the coefficients of other polynomials whose roots are products, differences, and quotients of the roots of p and q.

We need only a few facts about matrices. The n by n matrix

$$P=\begin{bmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ & & & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & & & & 1 \\ -p_n & & & \cdots & -p_1 \end{bmatrix}$$

is called the **companion matrix** [**3**, p. 19] of the polynomial $p(x)$; its characteristic polynomial is $\det(xI-P)=p(x)$. In order to see this, add x^{i-1} times the ith column of $xI-P$ to its first column, for $i=2,\ldots,n$. This process leaves the determinant of $xI-P$ unchanged and obviously equal to $p(x)$. So given any monic polynomial $p(x)$, there is a matrix having $p(x)$ as its characteristic polynomial.

Next suppose that P is an n by n matrix with eigenvalues $\lambda_1,\ldots,\lambda_n$. It is well known [**2**, p. 183] that P is similar to an upper triangular matrix having $\lambda_1,\ldots,\lambda_n$ on the main diagonal. Now let $f(x)=f_0x^t+f_1x^{t-1}+\cdots+f_t$ be a polynomial of degree t. A simple computation shows that the matrix

$$f(P)=f_0P^t+f_1P^{t-1}+\cdots+f_tI$$

is similar to an upper triangular matrix having $f(\lambda_1),\ldots,f(\lambda_n)$ on the main diagonal. Thus the eigenvalues of $f(P)$ are $f(\lambda_1),\ldots,f(\lambda_n)$, and the coefficients of $\det(xI-f(P))$ are polynomials in the coefficients $p_1,\ldots,p_n$ of $p(x)$.

Now the problem of finding the coefficients of a polynomial with roots $f(\lambda_1),\ldots,f(\lambda_n)$, where f is a polynomial over F, can be solved quite easily by means of the following summary of the previous two paragraphs:

THEOREM 1. *Let P be the companion matrix of $p(x)$. Then the polynomial $h(x)=\det(xI-f(P))$ has roots $f(\lambda_1),\ldots,f(\lambda_n)$, where $\lambda_1,\ldots,\lambda_n$ are the roots of $p(x)$.*

It is also possible to use this matrix method on more exotic choices for the function f. For example, f can be a rational function. In this case, the computation requires finding the inverse of a matrix. If $f(x)=n(x)/d(x)$ is a rational function and $d(\lambda_i)\neq 0$, for $i=1,\ldots,n$, then the polynomial $h(x)=\det(xI-(n(P))(d(P))^{-1})$ has roots $f(\lambda_1),\ldots,f(\lambda_n)$. It is even true that the polynomial $\det(xI-\exp(P))$ has roots $\exp(\lambda_1),\ldots,\exp(\lambda_n)$, where F is the real number field and f is the exponent function [**3**, p. 98, 99].

Theorem 1 has another interesting consequence. Recall that the resultant of $p(x)$ and $q(x)$ is a polynomial expression $R(p,q)$ in the coefficients of $p(x)$ and $q(x)$ that equals the common (!) value [**1**, p. 71] of the three products $p(\mu_1)\cdots p(\mu_m)$, $(-1)^{nm}q(\lambda_1)\cdots q(\lambda_n)$, and $\Pi_{i,j}(\lambda_i-\mu_j)$. From this definition we see that $R(p,q)=0$ if and only if $p(x)$ and $q(x)$ have a common root. We may use Theorem 1 to compute the resultant since by this theorem $\det(xI-p(Q))=(x-p(\mu_1))\cdots(x-p(\mu_m))$, where Q is the companion matrix of $q(x)$. Set $x=0$. Then $R(p,q)=p(\mu_1)\cdots p(\mu_m)=\det(p(Q))$. Similarly, $R(q,p)=\det(q(P))=(-1)^{nm}R(p,q)$. Each entry in the m by m matrix $p(Q)$ is of the form $a_0+p_1a_1+\cdots+p_na_n$ where $a_0,\ldots,a_n$ are expressions in the coefficients of $q(x)$ that do not involve any of the p_i. Thus the degree of $R(p,q)$ in the p_i is m. Similarly, the degree of $R(p,q)$ in the cofficients of $q(x)$ is n. The reader can compare this form of the resultant with some previous ones involving determinants [**1**, p. 74–82; **3**, p. 84].

Theorem 1 can also be used to compute the discriminant $D(p)$ of a polynomial $p(x)$ since $D(p)=\det p'(P)$ [1], where P is the companion matrix of p, and p' denotes the derivative of p. As an example, we shall find the discriminant of the general cubic polynomial $p(x)=x^3+p_1x^2+p_2x+p_3$:

$$\begin{aligned} D(p) &= \det(p'(P)) = \det(3P^2+2p_1P+p_2I) \\ &= \det\begin{bmatrix} p_2 & 2p_1 & 3 \\ -3p_3 & -2p_2 & -p_1 \\ p_1p_3 & p_1p_2-3p_3 & p_1^2-2p_2 \end{bmatrix} \\ &= 4p_1^3p_3 - p_1^2p_2^2 - 18p_1p_2p_3 + 4p_2^3 + 27p_3^2. \end{aligned}$$

Now we turn to the problem of finding polynomials with roots $\lambda_i\mu_j$, $\lambda_i-\mu_j$ and λ_i/μ_j, $i=1,\ldots,n$, $j=1,\ldots,m$. Again let Q be the companion matrix of $q(x)$, and let $p^{(i)}(x)$ denote the ith derivative of $p(x)$.

THEOREM 2. *The polynomials $h(x), k(x), d(x)$ where*

(1) $$h(x)=\det(x^nI+p_1x^{n-1}Q+\cdots+p_nQ^n)$$

(2) $$\begin{aligned} k(x) &= \det(p(xI+Q)) \\ &= \det\left(x^n\frac{p^{(n)}(Q)}{n!}+\cdots+x^2\frac{p^{(2)}(Q)}{2}+xp^{(1)}(Q)+p(Q)\right) \end{aligned}$$

(3) $$d(x)=\det(p(xQ))$$

have the roots $\lambda_i\mu_j$, $\lambda_i-\mu_j$, and λ_i/μ_j, respectively, for $i=1,\ldots,n$, $j=1,\ldots,m$.

Proof. From $x^n+p_1x^{n-1}+\cdots+p_n=(x-\lambda_1)\cdots(x-\lambda_n)$, it follows that

(4) $$x^n+p_1x^{n-1}y+\cdots+p_ny^n=(x-\lambda_1y)\cdots(x-\lambda_ny).$$

We can prove that all three parts of Theorem 2 by making appropriate substitutions for x and y in equation (4). First, substitute Q for y in equation (4). Then

$$\begin{aligned} h(x) &= \det(x^nI+p_1x^{n-1}Q+\cdots+p_nQ^n) \\ &= \det(xI-\lambda_1Q)\cdots(xI-\lambda_nQ) \\ &= \det(xI-\lambda_1Q)\cdots\det(xI-\lambda_nQ). \end{aligned}$$

The eigenvalues of λ_iQ are $\lambda_i\mu_1,\ldots,\lambda_i\mu_m$. Thus the roots of $h(x)$ are $\lambda_i\mu_j$ where $i=1,\ldots,n$; $j=1,\ldots,m$.

Similarly, in order to prove the second part of the theorem, we substitute $xI+Q$ for x and I for y in equation (4). The third part of the theorem is proved by substituting xQ for x and I for y in equation (4).

We remark, in conclusion, that this theorem has as a corollary the fact that the set of algebraic integers is closed under multiplication and subtraction (and hence addition). Recall that a complex number is an **algebraic integer** if it is a root of a monic polynomial with coefficients that are ordinary integers. Suppose that λ and μ are algebraic integers and that they are roots of the monic polynomials $p(x)$ and $q(x)$, respectively. Then $\lambda\mu$ is a root of the monic polynomial $h(x)$ in (1). Since determinants and matrix multiplication involve only addition and multiplication, $h(x)$ has integral coefficients. A similar argument using the polynomial $k(x)$ in (2) proves that $\mu-\lambda$ is also an algebraic integer. Theorem 2 gives a way to actually compute these polynomials. For example, $\sqrt{2}$ and $\omega=\exp(2\pi i/3)$ are algebraic integers. These numbers are roots of the polynomials $p(x)=x^2-2$ and $q(x)=x^2+x+1$, respectively. In order to find a monic polynomial $h(x)$ such that $h(\sqrt{2}\omega)=0$, we compute, using formula (1),

$$h(x) = \det(x^2 I - 2Q^2) = \det \begin{bmatrix} x^2+2 & 2 \\ -2 & x^2 \end{bmatrix} = x^4 + 2x^2 + 4.$$

Thus $\sqrt{2}\omega$ is a root of $x^4 + 2x^2 + 4$. To find a polynomial $k(x)$ such that $k(\sqrt{2} - \omega) = 0$, we compute, using formula (2),

$$\begin{aligned} k(x) &= \det \left(x^2 \frac{p^{(2)}(Q)}{2} + xp^{(1)}(Q) + p(Q) \right) \\ &= \det(x^2 I + 2xQ + (Q^2 - 2I)) \\ &= \det \begin{bmatrix} x^2-3 & 2x-1 \\ -2x+1 & x^2-2x-2 \end{bmatrix} \\ &= x^4 - 2x^3 - x^2 + 2x + 7. \end{aligned}$$

Thus $\sqrt{2} - \omega$ is a root of $x^4 - 2x^3 - x^2 + 2x + 7$.

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Spectra of Numbers

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For a positive real number α, the sequence $S(\alpha)$, sometimes called the **spectrum** of α, is defined by

$$S(\alpha) = ([\alpha], [2\alpha], [3\alpha], \ldots)$$

where $[x]$ denotes the greatest integer $\leqq x$. The spectra of real numbers are known to have many interesting properties (cf. [7], [1], [2], [3], [9]). For example, for any three spectra $S(\alpha_1), S(\alpha_2), S(\alpha_3)$, some pair of them must have infinitely many elements in common.

In working with spectra, it is often useful to know whether a particular finite sequence is the beginning of some spectrum and, if so, what its possible continuations are. In this note we characterize the initial segments of spectra as sequences which are "nearly" linear. As a result, this characterization allows us to answer both of the preceding questions rather efficiently.

Let us call a finite sequence $A = (a_1, a_2, \ldots, a_n)$ **nearly linear** if all the following inequalities hold:

(1) $\max\{a_i + a_{k-i}: 1 \leqq i < k\} \leqq a_k \leqq 1 + \min\{a_i + a_{k-i}: 1 \leqq i < k\}$, for $1 < k \leqq n$.

For example, the sequences (2, 4, 6, 8, 11, 13, 15, 17, 20) and (1, 3, 4, 6, 7, 9, 11, 12, 14, 15, 17, 18) are nearly linear. Note that if α is an integer then $A = S(\alpha)$ is actually linear and $a_i + a_{k-i} = a_k$ for $1 \leqq i < k \leqq n$, and (1) holds trivially. Moreover, the initial segment $S_n(\alpha) = \{[\alpha], [2\alpha], \ldots, [n\alpha]\}$ of the spectrum of each real number α is also nearly linear: if $1 \leqq i < k \leqq n$, then $[k\alpha]$ equals either $[i\alpha] + [(k-i)\alpha]$ or $[i\alpha] + [(k-i)\alpha] + 1$, and so (1) holds. The main result of this note is the converse of this result.

THEOREM. *If $A = (a_1, a_2, \ldots, a_n)$ is nearly linear, then A is the initial segment $S_n(\alpha)$ for some α.*

Proof. The proof will be by induction on n. The result is clear for $n = 1$. Suppose the theorem holds for all values less than some integer $n > 1$ and let $A = (a_1, \ldots, a_n)$ be nearly linear. Since the sequence $A' = (a_1, \ldots, a_{n-1})$ is also nearly linear then by the induction hypothesis $A' = S_{n-1}(\alpha)$ for some α. This implies $a_k \leqq k\alpha < a_k + 1$ for $1 \leqq k < n$, which is equivalent to

$$L = \max_{1 \leqq k < n} \frac{a_k}{k} \leqq \alpha < \min_{1 \leqq k < n} \frac{a_k + 1}{k} = U.$$

The theorem will be proved if we can show

$$L^* = \max\left(L, \frac{a_n}{n}\right) < \min\left(U, \frac{a_n + 1}{n}\right) = U^*, \tag{2}$$

for then the desired α can be found just by choosing $L^* \leqq \alpha < U^*$.

There are two possibilities. First suppose $L^* = a_n/n$. By the definition of U, $U = (a_d + 1)/d$ for some d, $1 \leqq d < n$. By hypothesis, $a_{n-d}/(n-d) < (a_d + 1)/d$. Therefore, since A is nearly linear (use (1) with $k = n$),

$$L^* = \frac{a_n}{n} \leqq \frac{(a_d + 1) + a_{n-d}}{d + (n-d)} < \frac{a_d + 1}{d} = U$$

and consequently (2) holds.

Now, suppose $L^* = L$. Then for some d', $1 \leqq d' < n$, $L = a_{d'}/d'$. By hypothesis, $a_{d'}/d' < (a_{n-d'} + 1)/(n-d')$. Therefore, as above,

$$L = \frac{a_{d'}}{d'} < \frac{a_{d'} + (a_{n-d'} + 1)}{d' + (n-d')} \leqq \frac{a_n + 1}{n}$$

and, again, (2) holds. This proves the theorem.

In principle, a similar characterization should exist for nonhomogeneous spectra, i.e., sequences of the form $S(\alpha, \beta) = ([\alpha + \beta], [2\alpha + \beta], [3\alpha + \beta], \ldots)$. However, because it is possible for $[p\alpha + \beta] + [q\alpha + \beta]$ to equal $[(p+q)\alpha + \beta] - 1$, $[(p+q)\alpha + \beta]$ or $[(p+q)\alpha + \beta] + 1$, the conditions corresponding to (1) in this case will necessarily be more complicated (cf. [**8**], [**6**], [**10**]). In general, a much wider variety of behavior is possible using nonhomogeneous spectra. For example, in contrast to the intersection result (mentioned in the first paragraph) which holds for any three (homogeneous) spectra, it is possible to have infinitely many nonhomogeneous spectra which are pairwise mutually disjoint, e.g., $S(2^n, 2^{n-1})$, $n = 1, 2, 3, \ldots$.

We conclude with an interesting open question concerning nonhomogeneous spectra. An old result states that it is possible to partition the positive integers $\mathbb{Z}^+$ into two disjoint (homogeneous) spectra but that such a partition is impossible for three or more spectra (cf. [**4**], [**5**], [**7**], [**9**]). However for any n, $\mathbb{Z}^+$ can be partitioned into n mutually disjoint nonhomogeneous spectra $S(\alpha_i, \beta_i)$, $1 \leqq i \leqq n$. Such decompositions are completely understood in the case that at least one α_k is irrational (cf. [**6**]); basically, all the $S(\alpha_i, \beta_i)$ are generated from some trivial decomposition of the form $\mathbb{Z}^+ = S(\alpha) \cup S(\beta)$ with $(1/\alpha) + (1/\beta) = 1$. On the other hand, if all the α_i are rational, then the situation is much less clear. The following striking conjecture has been proposed by Aviezri Fraenkel: If all the α_i are distinct and rational and $n \geqq 3$, then $\{\alpha_i : 1 \leqq i \leqq n\} = \{(2^n - 1)/2^i : 0 \leqq k < n\}$.

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The Psi Function

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The gamma function defined for $x > 0$ by $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ is well known as a generalized factorial because it satisfies the recurrence formula $\Gamma(x+1) = x\Gamma(x)$. It follows that $\Gamma(n) = (n-1)!$ for each positive integer n. The lesser known psi function, the subject of this paper, is defined as the logarithmic derivative of the gamma function:

$$\psi(x) = \frac{d}{dx}\ln\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

(The psi function and its derivatives ψ', ψ'', ψ''', etc., are also known as the digamma, trigamma, tetra-gamma, and pentagamma functions. Collectively they are called the polygamma functions.)

The integral which defines the gamma function converges for all $x > 0$. However the recurrence formula $\Gamma(x+1) = x\Gamma(x)$ makes it possible to extend the definition of $\Gamma(x)$ to negative non-integral values of x. This extension means that the entire family of polygamma functions (that is, $\psi, \psi', \ldots, \psi^{(n)}, \ldots$) is defined for all real numbers except $0, -1, -2, \ldots$.

A recurrence formula for the psi function can be obtained by finding the logarithmic derivative of $\Gamma(x+1)$ in two different ways and then equating the results. The recurrence formula for $\Gamma(x)$ yields

$$\frac{d}{dx}\ln\Gamma(x+1) = \frac{d}{dx}[\ln x + \ln\Gamma(x)]$$
$$= \frac{1}{x} + \frac{\Gamma'(x)}{\Gamma(x)} = \frac{1}{x} + \psi(x),$$

whereas the definition of the psi function yields

$$\frac{d}{dx}\ln\Gamma(x+1) = \frac{\Gamma'(x+1)}{\Gamma(x+1)} = \psi(x+1).$$

Equating these results gives the fundamental difference equation of the psi function:

$$\psi(x+1)-\psi(x)=\frac{1}{x}. \tag{1}$$

Its nth derivative yields a corresponding equation for the polygamma function:

$$\psi^{(n)}(x+1)-\psi^{(n)}(x)=\frac{(-1)^n n!}{x^{n+1}}. \tag{2}$$

(The derivation of these recurrence relations is troublefree only for $x>0$; but their validity can be extended to the non-integral negative x by more careful methods.)

The special value $\psi(1)$, which we will use frequently, can be shown to equal the negative of Euler's constant [1]: $\psi(1)=\Gamma'(1)/\Gamma(1)=\Gamma'(1)=-\gamma=-0.5772156649\ldots$. Mathematicians have as yet been unable to determine whether Euler's constant is an irrational or rational number. Nonetheless, it plays a crucial role in the evaluation of the psi function. For example, repeated use of the recurrence relation (1) yields

$$\psi(5)=\frac{1}{4}+\psi(4)=\frac{1}{4}+\left[\frac{1}{3}+\psi(3)\right]=\cdots$$

$$=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}-\gamma=1.5061176684\ldots.$$

In general, for positive integral arguments,

$$\psi(n+1)=\sum_{i=1}^{n}\frac{1}{i}-\gamma,\ \text{for } n=1,2,\ldots. \tag{3}$$

When the argument is non-integral, the recurrence formula (1) reduces the evaluation to the range [1, 2]. For example,

$$\psi(6.38)=\frac{1}{5.38}+\frac{1}{4.38}+\frac{1}{3.38}+\frac{1}{2.38}+\frac{1}{1.38}+\psi(1.38);$$

Tables give $\psi(1.38)=-0.08209\ldots$, so $\psi(6.38)=1.77275\ldots$.

Values of the psi function for small arguments can be computed from a special infinite series which is derived from an infinite product representation of the gamma function [2]:

$$\Gamma(x)=\frac{1}{x}e^{-\gamma x}\prod_{i=1}^{\infty}\frac{e^{x/i}}{1+(x/i)}.$$

(In reciprocal form this formula is called Weierstrass's formula, although it was first developed by Francis. W. Newman in 1848.) Taking the logarithm of both sides yields

$$\ln\Gamma(x)=-\ln x-\gamma x-\sum_{i=1}^{\infty}\left[\ln\left(1+\frac{x}{i}\right)-\frac{x}{i}\right],$$

a uniformly convergent infinite series representing $\ln\Gamma(x)$. Term-by-term differentiation produces

$$\frac{d}{dx}\ln\Gamma(x)=\psi(x)=-\frac{1}{x}-\gamma+\sum_{i=1}^{\infty}\frac{x}{i(x+i)}. \tag{4}$$

So, for example,

$$\psi(1.38)=-\frac{1}{1.38}-0.577215664\ldots+\sum_{i=1}^{\infty}\frac{1.38}{i(1.38+i)}.$$

Unfortunately, the series on the right does not converge very rapidly and requires almost 200 terms for six decimal accuracy.

Suppose it is required to compute a psi function of large argument, say $\psi(50{,}000)$. The generalized recurrence formula (3) is not very useful because of the slow (yet divergent) growth of the harmonic series Σi^{-1}. We need, instead, an asymptotic expansion. Such an expansion is a divergent series with the useful property that the terms first decrease in size and then begin to increase. If the series is truncated at the point where the terms begin to increase in size, the asymptotic series provides a good approximation. An asymptotic expansion useful for computing a psi function of large argument is [**4**, p. 259]

$$\psi(x)\sim \ln x-\frac{1}{2x}-\sum_{n=1}^{\infty}\frac{B_{2n}}{(2n)x^{2n}}=\ln x-\frac{1}{2x}-\left[\frac{1}{12x^2}-\frac{1}{120x^4}+\frac{1}{252x^6}-\cdots\right], \tag{5}$$

where the coefficients B_{2n} are the Bernoulli numbers: $B_2=1/6$, $B_4=-1/30$, $B_6=1/42$. These numbers eventually become very large, so the series diverges. Nonetheless, we can estimate $\psi(50{,}000)$ rather accurately by using only the first few terms:

$$\psi(50{,}000)\sim \ln 50{,}000-\frac{1}{100{,}000}-\frac{1}{12(50{,}000)^2}+\cdots=10.81976828\ldots.$$

Historically the first example of an asymptotic series expansion was one, similar to the one given above, that was used for computing the gamma function. The role of asymptotic series in computation contrasts sharply with the central role of convergence in analysis. While mathematicians find the remainder terms of an infinite series far more important than a finite number of terms at the beginning of series, scientific workers seek series whose first few terms decrease in magnitude rapidly, and care little whether the series ultimately converges or diverges. It has become the rule in table making to look for an asymptotic expansion first.

The recurrence formula for the polygamma functions, when turned around, yields a powerful instrument for computation of sums of the form $\Sigma 1/x^n$. For instance, to estimate the sum of the reciprocals of the first 50,000 integers, write (3) as

$$\begin{aligned}\sum_{i=1}^{50{,}000}\frac{1}{i}&=\psi(50{,}001)+\gamma\\&=\psi(50{,}000)+\frac{1}{50{,}000}+\gamma\\&=10.81976828\ldots+.00002+0.57721566\ldots\\&=11.39700394\ldots.\end{aligned}$$

(This example shows dramatically how slowly the harmonic series diverges). Similar analysis with (2) shows for instance, that

$$\sum_{i=1}^{100}\frac{1}{i^2}=\psi'(1)-\psi'(101). \tag{6}$$

The special value $\psi'(1)$ is known to be $\pi^2/6=1.64493406\ldots.$ To compute $\psi'(101)$ we write the recurrence relation (2) as $\psi'(101)=\psi'(100)-[1/10{,}000]$. Because the derivative of this asymptotic series is still an asymptotic series, we can estimate $\psi'(101)$ as we did $\psi(50{,}001)$. The derivatve of (5) yields:

$$\psi'(101)\sim -\frac{1}{10{,}000}+\frac{1}{100}+\frac{1}{2(100)^2}+\frac{1}{6(100)^3}-\frac{1}{30(100)^5}+\cdots=0.00995016\ldots.$$

Thus

$$\sum_{i=1}^{100}\frac{1}{i^2}=1.644934\ldots-0.009950\ldots=1.63498\ldots.$$

(Of course, the infinite series Σi^{-2} is well known, and equals $\pi^2/6 = \psi'(1) = 1.64493406\ldots$.) The methods used in this paragraph (especially equation (6)) can be easily generalized to higher powers using the polygamma functions:

$$(-1)^n(n-1)! \sum_{i=1}^{k} \frac{1}{i^n} = \psi^{(n-1)}(1) - \psi^{(n-1)}(k+1).$$

If we combine (1) and (4) we obtain another formula useful in summing certain infinite series:

$$\sum_{i=1}^{\infty} \frac{x}{i(x+i)} = \gamma + \psi(x+1). \tag{7}$$

So, for example, to find the sum of $\Sigma_{i=1}^{\infty}[i(3i-1)]^{-1}$ we express the general term in the form required for (7) and obtain

$$\sum_{i=1}^{\infty} \frac{1}{i(3i-1)} = -\sum_{i=1}^{\infty} \frac{-\frac{1}{3}}{i(-\frac{1}{3}+i)} = -[\gamma + \psi(\tfrac{2}{3})] = -0.5772\ldots + 1.3196\ldots = 0.742\ldots.$$

More complicated sums can be handled similarly by means of partial fractions. For example, to find the sum $S = \Sigma_{i=1}^{\infty}[(i+1)(2i+1)(4i+1)]^{-1}$, we express the general term in its partial fraction expansion

$$\frac{1}{3(i+1)} - \frac{2}{2i+1} + \frac{8}{3(4i+1)}.$$

Then

$$S = \sum_{i=1}^{\infty} \left(\frac{\frac{1}{3}}{i+1} - \frac{1}{i+\frac{1}{2}} + \frac{\frac{2}{3}}{i+\frac{1}{4}} \right) = [\tfrac{1}{3}\psi(2) - \psi(\tfrac{3}{2}) + \tfrac{2}{3}\psi(\tfrac{5}{4})] = 0.047198\ldots.$$

More generally, if u_i is a rational fraction where the numerator is a polynomial of degree less than the number m of factors in the denominator, then $\Sigma_{i=1}^{\infty} u_i$ can be expressed as a finite series of psi functions [5]

$$\sum_{i=1}^{\infty} u_i = -\sum_{k=1}^{m} A_k \psi(1+x_k)$$

where the A_k are the constants that appear in the partial fraction expansion of u_i.

This short survey is meant to be suggestive rather than definitive. Much material has been omitted and plausible demonstrations have replaced complete proofs. (A good general reference, which inspired this paper, is [5].) Our examples have, we hope, shed some light upon parts of the subject called the sum calculus and on the value of tables for scientific calculation. We have not treated specific applications. (One of the more interesting is in electrical circuit theory [3]: If we take resistors $R_n = 1/n$ in parallel with inductances $L_n = 1/n^2$, and connect the circuits corresponding to $n = 1, 2, 3, \ldots$, in series, then the impedance of the network will be $\psi(p) + \gamma$ where p is an oscillation constant.) The material in this paper is not new, but it remains useful in many contexts. There are, in fact, several possible avenues for further development. For example, we have no general means for summing series of the form $\Sigma 1/x^{5/3}$. Perhaps it is possible to replace n in (2) with ν where ν is a fraction, and then compute $\psi^{(\nu)}(x)$ by means of the Riemann-Liouville operator in the fractional calculus.

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Solutions via Group Theory for Linear Diophantine Equations

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In this note we give an example of the interplay of group theory and elementary number theory. Not too surprisingly the group $Z^n = \Sigma_{i=1}^n Z$ of n-tuples of integers under addition is a convenient tool for use in solving the following problem: given integers $a_1, \ldots, a_n$ and c, determine the set S_n of all integral solutions of the diophantine equation $a_1x_1 + \cdots + a_nx_n = c$. This problem is well known, and many similarities exist between the approach offered here and the number theoretic technique given in [3]. A different technique utilizing matrices is given in [1] and [2].

The existence of integral solutions to our problem is completely determined by the following well-known condition: Let d be the greatest common divisor of the integers $a_1, \ldots, a_n$, and let $a_i = d\bar{a}_i$, for $i = 1, \ldots, n$. Then the set of solutions S_n to the diophantine equation $a_1x_1 + \cdots + a_nx_n = c$ is non-empty if and only if d divides c, in which case S_n coincides with the set $\bar{S}_n$ of integral solutions for the equation $\bar{a}_1x_1 + \cdots + \bar{a}_nx_n = \bar{c}$ where $c = d\bar{c}$. This reduces our problem to consideration of a linear diophantine equation with relatively prime coefficients.

To actually determine an integral solution in case $c = d\bar{c}$ we proceed as follows. Since d is the greatest common divisor of $a_1, \ldots, a_n$, integers $b_1, \ldots, b_n$ may be found (by reversing the steps in the computation of the greatest common divisor of $a_1, \ldots, a_n$ via the Euclidean Algorithm) such that $a_1b_1 + \cdots + a_nb_n = d$. It follows that $s = (b_1\bar{c}, \ldots, b_n\bar{c}) = \bar{c}(b_1, \ldots, b_n)$ will be a particular solution for $a_1x_1 + \cdots + a_nx_n = c$.

We now observe that if s and s' are both solutions for the equation $a_1x_1 + \cdots + a_nx_n = c$ then $s' - s$ will be a solution for the homogeneous equation $a_1x_1 + \cdots + a_nx_n = 0$. Moreover, the solution set H_n for the homogeneous equation $a_1x_1 + \cdots + a_nx_n = 0$ is a subgroup of the additive group of all ordered n-tuples of integers Z^n. Thus to find all (integral) solutions S_n for the linear diophantine equation $a_1x_1 + \cdots + a_nx_n = c$ we need only find the group of solutions H_n for the associated homogeneous equation where in fact we may assume the coefficients are relatively prime. Now solutions for homogeneous linear equations in two variables are easily determined: If a_1 and a_2 are relatively prime integers, then the solution set H_2 for the homogeneous linear diophantine equation $a_1x_1 + a_2x_2 = 0$ is the cyclic subgroup of Z^2 generated by $(-a_2, a_1)$. This result provides the foundation for an inductive procedure that yields the solution for the general n-dimension problem.

THEOREM. *Let $a_1, \ldots, a_n, a_{n+1}$ be relatively prime integers. Let $d = \gcd\{a_1, \ldots, a_n\}$ and $a_i = d\bar{a}_i$ for each $1 \leqq i \leqq n$. Let H_n be the solution set for $a_1x_1 + \cdots + a_nx_n = 0$; and H_{n+1} the solution set for $a_1x_1 + \cdots + a_nx_n + a_{n+1}x_{n+1} = 0$. If B_n is a generating set for the group H_n and $(m_1, \ldots, m_n)$ is a particular solution for the equation $\bar{a}_1x_1 + \cdots + \bar{a}_nx_n = -a_{n+1}$, then H_{n+1} is generated by $(m_1, \ldots, m_n, d)$ and the set $\{(x_1, \ldots, x_n, 0) \mid (x_1, \ldots, x_n) \in B_n\}$.*

Proof. We begin by rewriting the equation $a_1x_1 + \cdots + a_nx_n + a_{n+1}x_{n+1} = 0$ in the form $a_1x_1 + \cdots + a_nx_n = -a_{n+1}x_{n+1}$. Now $\bar{a}_1dm_1 + \cdots + \bar{a}_ndm_n = a_1m_1 + \cdots + a_nm_n = -a_{n+1}d$ by the choice of $(m_1, \ldots, m_n)$. So $k(m_1, \ldots, m_n, d) \in H_{n+1}$ for any k. In fact, since d and a_{n+1} are relatively prime, any solution of $a_1x_1 + \cdots + a_nx_n = -a_{n+1}x_{n+1}$ must have the property that $x_{n+1} = dk$ for some integer k. To complete our proof we focus on the equivalent equation $\bar{a}_1x_1 + \cdots + \bar{a}_nx_n = -a_{n+1}k$ whose solution set is given by $k(m_1, \ldots, m_n) + H_n$. From this it follows that

$$H_{n+1} = \bigcup_{k \in Z} \{(b_1, \ldots, b_n, b_{n+1}) \mid b_{n+1} = dk \text{ and } (b_1, \ldots, b_n) \in k(m_1, \ldots, m_n) + H_n\}.$$

This subgroup is generated by $(m_1,\ldots,m_n,d)$ and $\{(b_1,\ldots,b_n,0)\,|\,(b_1,\ldots,b_n)\in H_n\}$.

To illustrate our results, we consider the following question motivated by question B-1 on the 1975 Putnam examination. Let G be the subgroup of Z^2 generated by $\{(3,8),(4,-1),(5,4)\}$, let K be the subgroup of Z^2 generated by $(0,1)$ and let $B=\{(1,b)\,|\,b\in Z\}$. Find $G\cap K$ and $G\cap B$. We first observe that $G\cap K=\{(0,a)\,|\,(0,a)=x(3,8)+y(4,-1)+z(5,4)$ with $x,y,z\in Z\}$. So we must have $8x-y+4z=a$ and $3x+4y+5z=0$. Let H_3 denote the set of solutions of this last equation. By our results $H_3=\langle(-4,3,0),(1,-2,1)\rangle$. So x must be of the form $-4k+m$, y of the form $3k-2m$, and z equal to m for some integers m and k. This implies that a must be of the form $7(-5k+2m)$, which, since -5 and 2 are relatively prime, implies that a is simply any multiple of 7. So $G\cap K=\langle(0,7)\rangle$. A similar approach to the equations $6=8x-y+4z$ and $1=3x+4y+5z$ shows that $G\cap B=\{(1,b)\,|\,b\equiv 5\pmod 7\}$. This, of course, shows that $G=\langle(0,a),(1,b)\rangle$ if and only if $a=\pm 7$ and $b\equiv 5\pmod 7$.

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References

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Midpoint Solutions of $x^x = a^b$.

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One day while experimenting with my still relatively new SR–50 hand calculator, I wondered idly for how large x it would calculate x^x, and found the answer to be somewhere between 56 and 57. Since the largest calculated number it will display is just short of 10^{100}, I set about solving more precisely the equation $x^x=10^{100}$ and found the solution to 10 significant digits to be $x=56.96124843$.

In solving this problem, incidentally, I wrote the equation in the form $x=100/\log x$ (log will denote base 10 logarithm here) and discovered experimentally that starting with any reasonable guess as x_1, the iterations $x_{n+1}=100/\log x_n$ converged rather rapidly. To see why, I had to recall a bit of fixed point theory. We are looking for a fixed point of the function $f(x)=100/\log x$, as a mapping of $D=[10,100]$ into itself. It is well known that the iterations $x_{n+1}=f(x_n)$, starting from any x_1 in D, converge to a fixed point of f if f is a contraction; i.e., for some $M, 0<M<1$, $|f(x)-f(y)|\leqq M|x-y|$ for all x,y in D, and for this it suffices that $|f'(x)|\leqq M$ on D. (See Theorem 10, p. 116 in [1].) Now for our f, $|f'(x)|=|100(\log e)/x(\log x)^2|$, which is larger than 1 at the lower end of the interval. However, f in fact maps $[10,100]$ onto $[50,100]$, and on this subinterval it is a contraction. Thus for any choice of x_1, the term x_2 will fall between 50 and 100, and the theorem applies to the sequence beginning with x_2.

To return to my main theme, I was surprised how near the solution lay to 55, the midpoint of the interval from 10 to 100. This led me to ponder whether, for each a one can always find $b > a$ so that $x^x = a^b$ is satisfied by $x = (a+b)/2$, and if so, whether this b could be expressed explicitly or estimated as a function of a. Numerical experimentation (more fun with the SR–50) indicates that when b is near a, x is near the upper end of the interval $[a, b]$, and when b is very large relative to a, x is near the lower end. For example, the solution to $x^x = 100^{101}$ is approximately $x = 100.82$, in the top fifth of $[100, 101]$, while for $x^x = 10^{1,000,000}$ it is about $x = 189{,}481$, in the bottom fifth of the interval $[10, 1{,}000{,}000]$.

The first question is relatively easy to answer. Let $h(x) = a^x$, $f(x) = [(a+x)/2]^{(a+x)/2}$, and

$$r(x) = \ln \frac{h(x)}{f(x)} = x \ln a - \frac{a+x}{2}[\ln(a+x) - \ln 2].$$

Then $r'(x) = \ln a - \frac{1}{2}[1 + \ln(a+x) - \ln 2]$ decreases without bound. Since $r(a) = 0$ and, for $a > e$, $r'(a) = (\ln a - 1)/2 > 0$, there must be precisely one point $b > a$ for which $r(b) = 0$. Thus there is a unique $b > a$ for which

$$h(b) = a^b = \left(\frac{a+b}{2}\right)^{(a+b)/2} = f(b).$$

Since a determines b uniquely, we may write $b = g(a)$.

The second question is a bit more difficult, and the answer given below was suggested by more extensive numerical calculations, for which I finally resorted to heavier machinery. It turns out that b is asymptotic to $2a^2$, but that a better asymptotic formula is $b_1 = 2a^2 - a - 2a(\ln a)$.

To see this, we evaluate the function $r(x) = \ln[h(x)/f(x)]$ at the point b_1:

$$\begin{aligned} r(b_1) &= r(2a^2 - a - 2a \ln a) \\ &= a[(a - 1 - \ln a)\ln a - (a - \ln a)\ln(a - \ln a)] \\ &= a(a - \ln a)\ln\left[1 + \frac{\ln a}{a - \ln a}\right] - a \ln a \\ &< \frac{a(a - \ln a)\ln a}{a - \ln a} - a \ln a = 0. \end{aligned}$$

Since $r(x) < 0$ occurs only when $x > b$, we have $b < b_1$. A similar but slightly more delicate analysis applied to the point $b_2 = b_1 - \varepsilon a \ln a$ will show that $r(b_2) > 0$ for sufficiently large a (and correspondingly small ε). Thus, for such a, we will have $b_2 < b < b_1$.

To carry out this analysis we write $b_2 = 2a^2 - a - 2ka \ln a$ where $k = 1 + \varepsilon/2$. If $k < e$ (i.e., if $\varepsilon < 2(e-1)$) then the term $u = (k \ln a)/(a - k \ln a)$ is positive. Therefore $\ln(1+u) > u - u^2/2 = u(1 - (u/2))$. Hence

$$\begin{aligned} r(b_2) &= r(2a^2 - a - 2k\,a \ln a) \\ &= a[(a - 1 - k \ln a)\ln a - (a - k \ln a)\ln(a - k \ln a)] \\ &= a(a - k \ln a)\ln\left[1 + \frac{k \ln a}{a - k \ln a}\right] - a \ln a \\ &> a(a - k \ln a)\frac{k \ln a}{(a - k \ln a)}\left[1 - \frac{k \ln a}{2(a - k \ln a)}\right] - a \ln a \\ &= a \ln a\left[k - 1 - \frac{k^2 \ln a}{2(a - k \ln a)}\right]. \end{aligned}$$

The quantity in brackets is positive (and hence $r(b_2)>0$) exactly when $a/\ln a > \varepsilon^{-1}+2+3\varepsilon/4$. For $\varepsilon>0$ this is equivalent to $(3/4)\varepsilon^2 - d\varepsilon + 1 < 0$ where $d = a/\ln a - 2$. Both roots of this quadratic are real and positive when $d \geqq \sqrt{3}$, corresponding to $a \geqq 7.54$, approximately. The smaller root is $\varepsilon = 2/(d+\sqrt{d^2-3})$, which is clearly less than $2(e-1)$ — our other condition on ε — when $d \geqq \sqrt{3}$. We summarize these findings in the following statement:

THEOREM. *For every $a > e$ there is a unique $b = g(a)$ greater than a such that $a^b = [(a+b)/2]^{(a+b)/2}$. Moreover, if*

$$b_1 = g_1(a) = 2a^2 - a - 2a\ln a$$

$$b_2 = g_2(a) = 2a^2 - a - (2+\varepsilon)a\ln a$$

where

$$\varepsilon = \frac{2}{d+\sqrt{d^2-3}} \quad \textit{for} \quad d = \frac{a}{\ln a} - 2,$$

then $g_2(a) \leqq g(a) < g_1(a)$ for all a such that $d \geqq \sqrt{3}$.

a	b_2	b	b_1	$\Delta = b - b_2$
8	73.3624	79.5213	86.7289	6.1589
9	101.3794	105.7811	113.4500	4.4017
10	132.2026	135.8482	143.9483	3.6456
11	166.5478	169.7405	178.2463	3.1927
15	341.4961	343.8264	353.7585	2.3303
20	646.8855	648.7642	660.1707	1.8787
25	1049.772	1051.401	1064.056	1.629
30	1550.719	1552.183	1565.928	1.464
40	2848.041	2849.292	2864.890	1.251
60	6629.174	6630.187	6648.679	1.013
80	11997.25	11998.12	12018.88	.87
100	18955.56	18956.34	18978.97	.78
150	43319.88	43320.51	43346.81	.63
200	77651.01	77651.56	77680.67	.55
250	121957.36	121957.85	121989.27	.49
300	176243.90	176244.35	176277.73	.45
400	314769.82	314770.20	314806.83	.38
600	711681.87	711682.17	711723.68	.30
800	1268459.18	1268459.43	1268504.62	.25
1000	1985136.10	1985136.33	1985184.49	.23
2000	7967538.17	7967538.32	7967596.39	.15
4000	31929578.52	31929578.62	31929647.60	.08
7000	97868970.10	97868970.16	97869048.68	.06

TABLE 1

TABLE 1 illustrates how remarkably good our lower bound is as an approximation to $g(a)$. The data strongly suggest that $\Delta = b - b_1$ decreases monotonically to 0 as a tends to infinity; it would be interesting to try to prove this conjecture.

In closing, here is an entirely different challenge. With $a = 4$, there is a solution in integers, namely $4^{12} = 8^8$. It would be interesting to try to find other integral solutions, or even to find all of them.

Reference

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Tossing Coins Until All Are Heads

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Suppose n coins are tossed simultaneously. Those which come up heads are removed and those which show tails are tossed again. The process is continued until each of the coins has come up heads. How many (group) tosses will be required, on average, to accomplish this? The solution of this problem is almost routine. Nevertheless, it leads, as we will show in this note, to interesting and challenging complications when related to computer computation.

We begin with an easy computer simulation. The program listed in FIGURE 1 simulates 100 trials of the tossing of 1000 fair coins until no heads appear. The result of one simulation with this program is given in the following table where Y represents the number of tosses required to eliminate all tails:

Value of Y:	8	9	10	11	12	13	14	15	16	17
Frequency :	2	8	26	23	19	10	7	3	1	1

The data suggest that Y is not very large for 1000 coins and also that Y is not likely to deviate very much from its mean value (11.35 for this data), so that extreme values for Y are unusual.

To confirm this, we will obtain the probability distribution for the random variable Y. Consider first the random variable X denoting the number of tosses of a single coin necessary to obtain a head. The probability that X assumes the value x is given by the geometric distribution $P(X=x)=pq^{x-1}$, where p is the probability of a head and $q=1-p$ is the probability of a tail. Then the random variable Y, denoting the total number of (group) tosses necessary until each of n coins shows a head, is $\max(X_1, X_2, \ldots, X_n)$ where the X_i's are geometric random variables for each coin in the original group. To find the probability function for Y, we first find the distribution function $G(y)=P(Y\leqq y)=P\{\max(X_1,X_2,\ldots,X_n)\leqq y\}$. Now $\max(X_1,X_2,\ldots,X_n)\leqq y$ means that each of the X_i's is at most y. Since the X_i's are independent, $G(y)=[P(X\leqq y)]^n$, where X is a geometric random variable. Since

$$P(X\leqq y)=\sum_{x=1}^{y} pq^{x-1},$$

it follows that

$$G(y)=\left[\sum_{x=1}^{y} pq^{x-1}\right]^n.$$

Thus the probability function $g(y)=P(Y=y)$ is given by

$$\begin{aligned} g(y)=P(Y=y)&=P(Y\leqq y)-P(Y\leqq y-1)\\ &=\left[\sum_{x=1}^{y} pq^{x-1}\right]^n-\left[\sum_{x=1}^{y-1} pq^{x-1}\right]^n\\ &=(1-q^y)^n-(1-q^{y-1})^n. \end{aligned}$$

If there are two fair coins, for example, $g(y)=(2^{y+1}-3)/4^y$; this formula can easily be conjectured by considering some specific values for y.

The expected value $E(Y)$ of the number of tosses depends on the values of n and q. One can explore this dependence by running the simulation program for different numbers of coins and

```
10   RANDOMIZE
15   DIM X (1000)
20   FOR I = 1 to 100
30   N = 1000: Y = 0
40   C = 0: Y = Y + 1
50   FOR J = 1 to N
60   X(J) = RND
70   IF X(J) < = 0.5 GO TO 90
80   C = C + 1
90   NEXT J
100  IF C = 0 GO TO 120
110  N = C: GO TO 40
120  PRINT Y
130  NEXT I
140  END
```

TABLE 1

n	$E(Y)$
2	2.66667
3	3.14286
4	3.50476
5	3.79416
6	4.03482
7	4.24085
8	4.42108
9	4.58130
10	4.72556
15	5.28721
20	5.69044
25	6.00526
30	6.26355
35	6.48255
40	6.67263
45	6.84053
50	6.98633

TABLE 2

probabilities of a head. If q is held constant, this simulation shows that $E(Y)$ increases with n, but not rapidly, suggesting that a non-linear function is appropriate. Of course $E(Y)=\Sigma_{y=1}^{\infty} yg(y)$ by definition, and tables of this value may be calculated (with care) by computer. But it is also possible to find another expression for $E(Y)$ that is somewhat more explicit. We begin with the expression

$$g(y)=(1-q^{y})^{n}-(1-q^{y-1})^{n}=\sum_{x=1}^{n}\binom{n}{x}(-1)^{x+1}(1-q^{x})q^{xy-x},$$

obtained by expanding each term in $g(y)$ by the binomial theorem. Then

$$\begin{aligned} E(Y) &= \sum_{y=1}^{\infty}\sum_{x=1}^{n}\binom{n}{x}(-1)^{x+1}(1-q^{x})yq^{xy-x} \\ &= \sum_{x=1}^{n}\binom{n}{x}(-1)^{x+1}\frac{(1-q^{x})}{xq^{x}}\left(\sum_{y=1}^{\infty} xyq^{xy}\right) \\ &= \sum_{x=1}^{n}\binom{n}{x}(-1)^{x+1}(1-q^{x})^{-1} \\ &= \binom{n}{1}(1-q)^{-1}-\binom{n}{2}(1-q^{2})^{-1}+\binom{n}{3}(1-q^{3})^{-1}-\cdots+(-1)^{n+1}(1-q^{n})^{-1}. \end{aligned}$$

Values of $E(Y)$ for tossing n fair coins are given in TABLE 2.

Computations with this formula are likely to be misleading since the binomial coefficients $\binom{n}{k}$ become large very quickly while the factors $(1-q^{k})^{-1}$ rapidly approach 1. Unless one keeps an enormous number of significant places while computing, roundoff error may render the result meaningless. It is possible to combine some terms making use of the fact that $\binom{n}{k}=\binom{n}{n-k}$. For example

$$E(Y)=(1-q^{n})^{-1}+\sum_{i=1}^{[n/2]}\binom{n}{i}\left\{\frac{1}{1-q^{i}}-\frac{1}{1-q^{n-i}}\right\}(-1)^{i+1}$$

for n odd, with a similar result for n even. These formulas in fact were used to obtain the results in TABLE 2. But even with this simplification roundoff error will eventually undermine the accuracy of the result.

A graph of the data in TABLE 2 strongly suggests that $E(Y)$ is a logarithmic function of n. Indeed, if the number of coins being tossed is doubled, $E(Y)$ is increased approximately by 1; this is intuitively appealing. A simple least squares program gives the estimate $E(Y) \approx .952575 \log_2 n + 1.58995$, which is in remarkable agreement with the values in TABLE 2.

The problem of tossing a batch of n coins until all come up heads is an interesting one to simulate using the computer. While exact formulas for the probability function and expectation can be found, they are difficult to compute with and the programmer must be keenly aware of roundoff error. Least squares estimates for $E(Y)$ give good results for moderate values of n but eventually roundoff error constitutes the major portion of the result in either case. On the other hand, simulation is not as accurate as computation and is subject to the availability of large numbers of independent random numbers. Accurate solution to the coin tossing problem requires subtle balance of the several available means of computation.

Constructing Functions with Zero Moments

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Recall that (to use that most supercilious of introductions) the nth moment μ_n of a non-negative function $f(t)$, $0 \leqq t < \infty$, is defined by $\mu_n = \int_0^\infty t^n f(t)dt$, $n = 0, 1, \ldots$. More generally, if F defines a finite mass distribution on the positive axis, i.e., if F is a non-decreasing bounded function, the moments of F are defined by the Stieltjes integrals $\int_0^\infty t^n dF(t)$. (The reader who feels uneasy with these integrals will miss almost nothing if he restricts his attention to the case where F has a density f, or $dF(t) = f(t)dt$).

The so-called Stieltjes moment problem is this: given numbers μ_n, $n \geqq 0$, when does there exist a mass distribution with μ_n as its nth moment and when is the solution unique? Historically, the problem probably has its roots in probability theory where the moments of a probability distribution (a distribution with total mass one) play a crucial role. Although this is no place for a history lesson, suffice it to say that it is both important and true that a distribution having the same moments as the normal distribution *is* normal (here, moments are taken over the entire real axis). The reader with a historical bent can consult [**2**], especially p. 204.

That there exist distinct functions having identical moments was demonstrated by Stieltjes in 1894. He showed this in as thoroughly convincing a way as is possible, namely, by giving a concrete example (in fact, a real live formula). The reader might enjoy computing for him (her) self that if $f(t) = \exp(-t^{1/4})\sin(t^{1/4})$, then $\int_0^\infty t^n f(t)dt = 0$, $n \geqq 0$, and so the two densities $\exp(-t^{1/4})$ and $\exp(-t^{1/4})\ (1 - \sin(t^{1/4}))$ have the same moments. (If a snag develops, cheat and see [**4**], p. 126).

While one cannot help but be dazzled by this bit of mathematical prestidigitation, one is not left feeling terribly enlightened. The question remains, how do mortals go about producing such pairs of functions? The modest content of this note is an inductive method for doing just that. There is a slight bonus in this approach. It turns out that the construction permits the substitution of functions $\varphi_n(t)$ for t^n provided only that the φ_n behave like the powers of t in a very few respects. Specifically, suppose $\{\varphi_n(t)\}$, $n \geqq 1$, is a sequence of continuous functions defined for $t \geqq 0$ satisfying

(a) $\varphi_{n+1}(t) \geqq \varphi_n(t) > 0$ for $t \geqq t_0$,

and

(b) if $\Phi_n(x) = \int_{t_0}^x \varphi_n(t)dt$, then both $\Phi_n(x)$ and $\Phi_{n+1}(x)/\Phi_n(x)$ tend to $+\infty$ as $x \to \infty$.

The φ_n, in place of t^n, define a "φ-moment problem" in an obvious way.

THEOREM. *With φ_n as above, there is a non-zero step function f such that $\int_0^\infty \varphi_n(t)|f(t)|dt < \infty$ and $\int_0^\infty \varphi_n(t)f(t)dt = 0$, $n = 1, 2, \ldots$.*

In view of this, the φ-moment problem is not generally uniquely solvable. For example, $|f|$ and $|f| - f$ are distinct non-negative functions with the same φ-moments.

Proof. We are going to define numbers β_n, $n \geqq 1$, inductively and a function f satisfying

(i) $\beta_1 = t_0$ and $\beta_{n+1} - \beta_n \geqq 1$,

(ii) f is a step function on $[\beta_n, \beta_{n+1})$ taking no more than $n+1$ values (actually n for $n \geqq 2$),

(iii) $\int_0^{\beta_{n+1}} \varphi_k(t)f(t)dt = 0$ for $k = 1, 2, \ldots, n$, and

(iv) $\int_{\beta_n}^{\beta_{n+1}} \varphi_{n-1}(t)|f(t)|dt \leqq 2^{-n}$, for $n \geqq 2$.

Once this is accomplished, we'll be done, for (iv) and (a) imply

$$\int_0^\infty \varphi_n(t)|f(t)|dt = \int_0^{\beta_{n+1}} \varphi_n(t)|f(t)|dt + \sum_{k=n+1}^{\infty} \int_{\beta_k}^{\beta_{k+1}} \varphi_n(t)|f(t)|dt$$

$$\leqq \text{constant} + \sum_{k=n+1}^{\infty} \int_{\beta_k}^{\beta_{k+1}} \varphi_{k-1}(t)|f(t)|dt$$

$$\leqq \text{constant} + \sum_{k \geqq n+1} 2^{-k} < \infty.$$

Hence (iii) gives $\int_0^\infty \varphi_k(t)f(t)dt = \lim_{n\to\infty} \int_0^{\beta_n} \varphi_k(t)f(t)dt = 0$.

In the midst of our inductive argument we are going to have to solve a system of linear equations whose coefficient matrix M has as its i, jth entry $\int_{a_{j-1}}^{a_j} \varphi_i(t)dt$ where $i, j = 1, \ldots, n$ and $t_0 < a_0 < a_1 < \cdots < a_n$. To estimate its determinant, expand in terms of the cofactors $\{C_i\}$ of the last column to get

$$\det M = \sum_{i=1}^{n} C_i \int_{a_{n-1}}^{a_n} \varphi_i(t)dt = \sum_i C_i \Phi_i(a_n) + C$$

where the C_i's and C are functions of $a_0, \ldots, a_{n-1}$. If the principal cofactor $C_n \neq 0$, then (thinking of $a_0, \ldots, a_{n-1}$ as fixed), (a) implies

$$|\det M| \geqq |C_n| \Phi_n(a_n) - \sum_{i=1}^{n-1} |C_i| \Phi_i(a_n) - |C| \geqq |C_n| \Phi_n(a_n) - (\text{constant})\Phi_{n-1}(a_n) - |C|$$

and (b) implies that if a_n is large enough,

$$(*) \qquad |\det M| \geqq (\text{constant})\Phi_n(a_n) > 0.$$

It follows that given $a_0 \geqq t_0$, we can successively pick $a_1, \ldots, a_n$ so that (*) holds.

Starting the induction is trivial. Set $\beta_1 = t_0$, $\beta_2 = \beta_1 + 1$, $f(t) = 0$, $0 \leqq t < t_0$ and pick any non-zero 2-valued step function on $[\beta_1, \beta_2)$ such that $\int_{\beta_1}^{\beta_2} \varphi_1(t)f(t)dt = 0$. Assume $\beta_1, \ldots, \beta_n$ and $f(t)$ on $[0, \beta_n)$ have been defined so that all is well. Let $\int_0^{\beta_n} \varphi_n(t)f(t)dt = -B$. If $B = 0$, setting $\beta_{n+1} = \beta_n + 1$ and $f \equiv 0$ on $[\beta_n, \beta_{n+1})$ does the trick.

Now assume that $B \neq 0$. Using (*), set $\beta_n = a_0$ and pick $a_0 < a_1 < \cdots < a_n = \beta_{n+1}$ so that $|\det M| \geqq K\Phi(a_n)$ for all $a_n > A$. (Think of the a_i's as fixed for $i \leqq n-1$; a_n will vary as needed). We will define $f(t) = \xi_j$ for $a_{j-1} \leqq t < a_j$. Then for (iii) to hold, we must have

$$(**) \qquad \sum_{j=1}^{n} \xi_j \int_{a_{j-1}}^{a_j} \varphi_i(t)dt = \begin{cases} 0, & \text{if } i = 1, \ldots, n-1, \\ B, & \text{if } i = n. \end{cases}$$

If M_j is the minor of the n, jth element of M, then by expanding M_j along its last column, we get (as we did for M) for $j \leqq n-1$,

$$M_j = \sum_{i=1}^{n-1} D_i \int_{a_{n-1}}^{a_n} \varphi_i(t)dt$$

where the $D_i = D_i(a_0, \ldots, a_{n-1})$. Thus

$$|M_j| \leqq \sum_{i=1}^{n-1} |D_i| \Phi_i(a_n) \leqq K_j' \Phi_{n-1}(a_n).$$

Solving (**) by Cramer's rule we obtain for $j \leqq n-1$, $\xi_j = (-1)^{n+j} BM_j/\det M$; thus $|\xi_j| \leqq K_j \Phi_{n-1}(a_n)/\Phi_n(a_n)$. Moreover, $\xi_n = BM_n/\det M$, so $|\xi_n| \leqq K_n/\Phi_n(a_n)$.

Since the K_j's depend only on $a_0, \ldots, a_{n-1}$, the ξ_j are functions of a_n alone and all that remains is to see whether $a_n = \beta_{n+1}$ can be chosen so that (iv) holds. The affirmative answer follows from the inequalities

$$\int_{\beta_n}^{\beta_{n+1}} \varphi_{n-1}(t)|f(t)|\,dt = \sum_{j=1}^{n} |\xi_j| \int_{a_{j-1}}^{a_j} \varphi_{n-1}(t)dt$$

$$\leqq \sum_{j=1}^{n-1} K_j \frac{\Phi_{n-1}(a_n)}{\Phi_n(a_n)} \int_{t_0}^{a_{n-1}} \varphi_{n-1}(t)dt + \frac{K_n}{\Phi_n(a_n)} \int_{t_0}^{a_n} \varphi_{n-1}(t)dt \leqq K' \frac{\Phi_{n-1}(a_n)}{\Phi_n(a_n)}.$$

The proof is finished.

The reader whose curiosity may have been piqued by the moment problem will certainly want to read further, to explore not only how uniqueness can fail but also to learn how it can be ensured. The most elegant result affirming uniqueness is that of Carleman: if f has moments μ_n such that the series $\Sigma \mu_n^{-1/2n}$ diverges, then f is uniquely determined by these moments. I would issue to the curious reader the following caveat. The standard two references **[1]** and **[3]** are both very thorny going, definitely not undergraduate fare. Try **[4]**, Chapter III and especially **[5]**, Appendix II.

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Maximin Hedges

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The gist of bookmaking is that the bookmaker hedges his bets; that is, he makes a combination of bets in which risks offset each other. Details vary from situation to situation, but the essential idea is illustrated by a brief look at baseball betting. A "price line" on a game between the Yankees and the Royals is typically quoted as follows: "6 : 7, Royals favorite". This means that a bettor must lay \$7 on the favorite (for illustrative purposes, the Royals) for a chance to win \$5 or lay \$5 on the underdog for a chance to win \$6. The base value of \$5 is arbitrary but traditional.

The bookmaker's profit picture depends, among other things, on the proportion of money booked on each side of the game. To see that, let x be the proportion of money bet on the favorite, and $1-x$ on the underdog. Suppose the favorite wins. For each dollar handled by the bookmaker, he pays $5x/7$ dollars to the backers of the favorite and receives $1-x$ dollars from the backers of the underdog for a net payoff of

$$P_1 = (1-x) - \frac{5x}{7} = 1 - \frac{12x}{7}.$$

On the other hand, if the underdog wins, the bookmaker pays $(6/5)(1-x)$ to the backers of the underdog and receives x from the backers of the favorite for a net payoff of

$$P_2 = x - \frac{6}{5}(1-x) = \frac{11}{5}x - \frac{6}{5}.$$

The bookmaker's payoff is summarized by

$$P = \begin{cases} P_1 = 1 - \dfrac{12x}{7} & \text{if the favorite wins,} \\ \\ P_2 = \dfrac{11x}{5} - \dfrac{6}{5} & \text{if the underdog wins.} \end{cases}$$

We plot these equations in FIGURE 1(a). Notice that where these two lines meet, the bookmaker's payoff is independent of the outcome of the ball game. If 77/137 of the money (roughly 56.2%) is booked on the favorite, the bookmaker's profit amounts to 5/137 of each dollar booked (roughly 3.65%). This bookmaker's profit is commonly called the vigorish. Notice also that as long as between 6/11 and 7/12 of the money is booked on the favorite (roughly between 54.55% and 58.33%), the bookmaker can do no worse than break even.

Of course nothing prevents a client bettor from hedging his bets, that is, from splitting his money between the two teams. If he splits his money in the proportion of x on the favorite and $1-x$ on the underdog, his payoff is described by

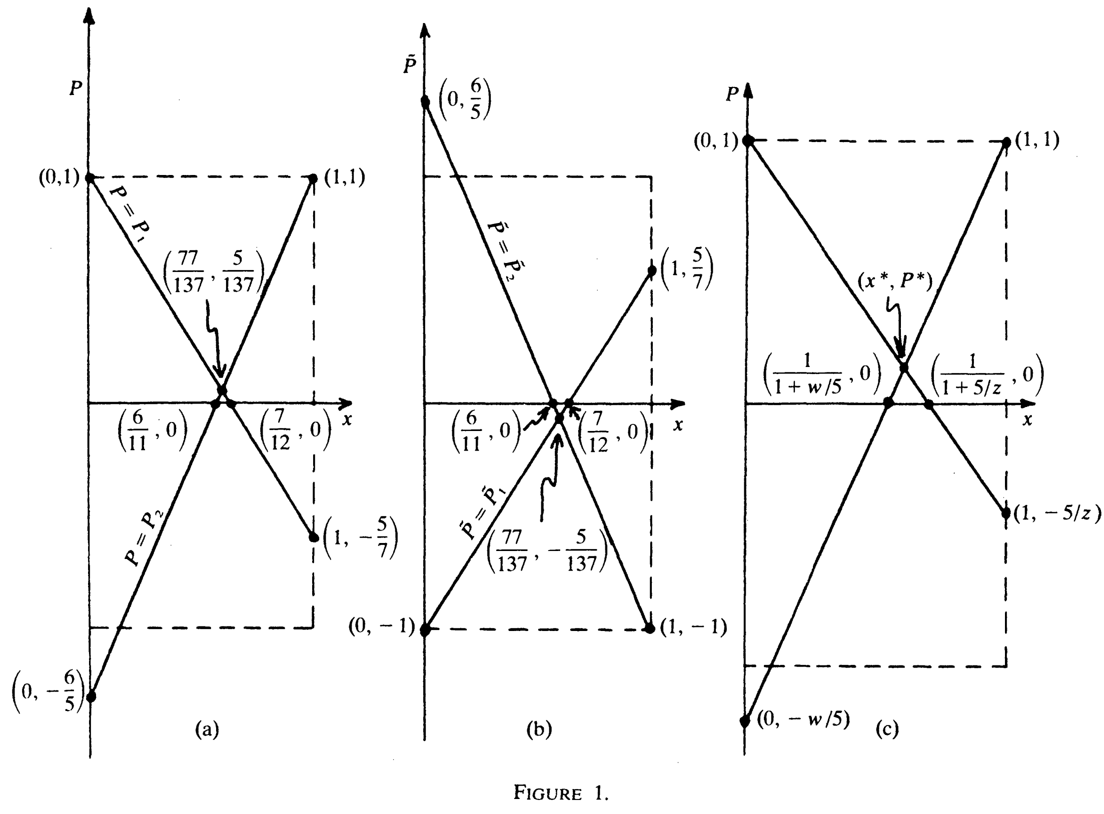

FIGURE 1.

$$\tilde{P} = \begin{cases} \tilde{P}_1 = \dfrac{12x}{7} - 1 & \text{if the favorite wins,} \\ \tilde{P}_2 = \dfrac{6}{5} - \dfrac{11x}{5} & \text{if the underdog wins.} \end{cases}$$

We plot these equations in FIGURE 1(b). Notice that the client's minimum payoff (like the bookmaker's) reaches its highest value when 77/137 of his commitment rides on the favorite. The difference is that this mix guarantees the client a sure *loss* amounting to 5/137 of the combined bet.

This situation is typical for the unfair odds that are the rule not only in sports betting, but in lotteries, casino games and racetrack betting. To insure a positive vigorish for the bookmaker with a price line of "$w : z$, Royals favorite," the odds must be set so that $z > w > 5$. In other words, bettors must pay more on the favorite (\$$z$ for a chance to win \$5) than the bookmaker will pay on the underdog (\$$w$ on a \$5 bet). The bookmaker's profit picture is then described by the following table:

	If the Favorite Wins	If the Underdog Wins
Backers of the Favorite (x at $5:z$)	$-\dfrac{5x}{z}$	x
Backers of the Underdog ($1-x$ at $w:5$)	$1-x$	$-\dfrac{w(1-x)}{5}$

So, overall, the bookmaker's payoff rate P (per dollar booked) is defined as follows:

$$P = \begin{cases} P_1 = 1 - x\left(1 + \dfrac{5}{z}\right) & \text{if the favorite wins,} \\ P_2 = \left(1 + \dfrac{w}{5}\right)x - \dfrac{w}{5} & \text{if the underdog wins.} \end{cases}$$

We plot these equations in FIGURE 1(c).

For any x, the bookmaker can do no worse than the minimum of P_1 and P_2; that minimum reaches its highest value when the two lines cross. This occurs at (x^*, P^*) where

$$x^* = \frac{1 + \frac{w}{5}}{\left(1 + \frac{5}{z}\right) + \left(1 + \frac{w}{5}\right)} \quad \text{and} \quad P^* = 1 - \frac{1}{\dfrac{1}{1+\frac{w}{5}} + \dfrac{1}{1+\frac{5}{z}}}.$$

So at (x^*, P^*) the bookmaker is guaranteed the highest minimum profit and this profit is independent of the outcome of the ball game. Note also that so long as x remains between $(1 + (5/z))^{-1}$ and $(1 + (w/5))^{-1}$ the bookmaker can do no worse that break even. For more examples of this and other types of sports bets the reader can consult [4].

We now consider a formulation of betting that includes sports betting as a special case. There are two sides to betting: one conventionally called the house and the other the client. Since the two sides play complementary games, we find it convenient to study betting from the viewpoint of the house; the client's viewpoint is obtained by a parallel argument.

The basic ingredient is an experiment (ball game, horse race, etc.) with a finite set of simple outcomes $B_1, B_2, \ldots, B_n$ that pay odds (to the client) of $w_1:1, w_2:1, \ldots, w_n:1$ respectively. The position of the house is stated as follows: if it loses B_i, it wins all B_j such that $j \neq i$. Now suppose money is booked on bets in a ratio of $x_i : 1$ on B_i, where $\Sigma_{i=1}^{n} x_i = 1$. For convenience put $X = (x_1, x_2, \ldots, x_n)$ and $W = (w_1, w_2, \ldots, w_n)$. The house's payoff on B_i is

$$P_i(X, W) = -x_i w_i + \sum_{j \neq 1} x_j = 1 - x_i(1 + w_i).$$

Overall, the house's payoff rate (per dollar handled) is $P_i(X, W)$ if B_i happens. So the worst the house can do is $\text{Min}_{1 \leq i \leq n} P_i(X, W)$. In bookmaking W is given and a key question is to find $X^* = (x_1^*, x_2^*, \ldots, x_n^*)$ that maximizes that minimum. We call X^* a **maximin hedge** for the house given W. The payoff yielded by such a hedge is denoted P^*. (A case where $n = 3$ is illustrated in FIGURE 2, where the house's payoffs for each outcome are superimposed on the same graph.)

The concept of maximin and its dual, minimax, was introduced by John von Neumann in connection with game theory and by Abraham Wald in connection with statistical decision theory. An accessible account dealing with both topics can be found in [**1**] and [**3**]. Although, in betting, both the house and the client can be viewed as playing a "game against nature", we do not assume any knowledge of the general theory in what follows. It turns out that the solution to the system of $n(n-1)$ equations $\{P_i(X, W) = P_j(X, W)\colon i, j = 1, 2, \ldots, n\}$ is the unique maximin hedge for W.

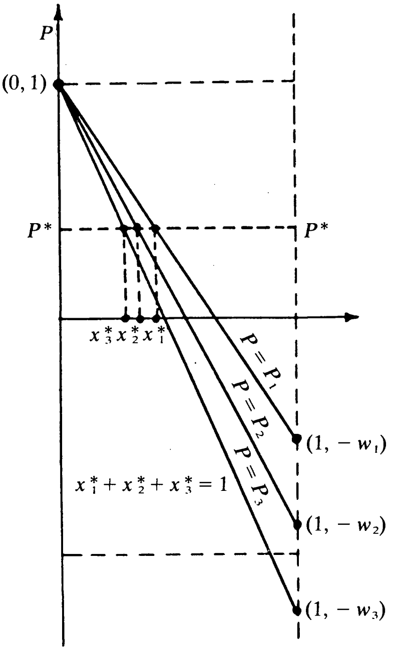

FIGURE 2.

To see this, we first solve the system of equations. To begin, solve

$$1 - x_i^*(1 + w_i) = 1 - x_j^*(1 + w_j)$$

for x_i^*:

$$x_i^* = x_j^* \left(\frac{1 + w_j}{1 + w_i}\right).$$

Then fix j and sum over i:

$$1 = x_j^*(1 + w_j) \sum_{i=1}^{n} \frac{1}{1 + w_i}.$$

Hence

$$x_j^* = \left[(1 + w_j) \sum_{i=1}^{n} \frac{1}{1 + w_i}\right]^{-1}, \quad j = 1, 2, \ldots, n.$$

Next, we show that X^* is the unique maximin hedge for W. Take any $X \neq X^*$. Since $\Sigma_{i=1}^{n} x_i = 1 = \Sigma_{i=1}^{n} x_i^*$, we must have $x_k > x_k^*$ for some k. Therefore, $P_k(X, W) < P_k(X^*, W)$, so

$$\underset{1 \leq i \leq n}{\text{Min}} \{P_i(X, W)\} \leq P_k(X, W) < P_k(X^*, W) = \underset{1 \leq i \leq n}{\text{Min}} \{P_i(X^*, W)\}.$$

So X^* is the unique maximin hedge for W; it yields a sure payoff (for the bookmaker) independent of the outcome of the experiment of

$$P^* = P_i(X^*, W) = 1 - \left[\sum_{i=1}^{n} \frac{1}{1 + w_i}\right]^{-1}.$$

Thus X^* yields a sure (positive) profit if and only if $\Sigma_{i=1}^{n}(1 + w_i)^{-1} > 1$. Note also that $\text{Min}_{1 \leq i \leq n}\{P_i(X, W)\} > 0$ for every X such that $x_i < [1 + w_i]^{-1}$, $i = 1, 2, \ldots, n$. (The results for our baseball example ($n = 2$) at the beginning of the paper follow immediately by putting $w_1 = 5/7$ and $w_2 = 6/5$.)

Parimutuel betting, seen from the house's viewpoint, presents precisely the reverse problem from that of bookmaking: the payoff odds are fully settled only after all the money is in; they can be set to yield any predetermined payoff rate. All the money bet on the same experiment (say, all the money bet "to win" in a given race) is pooled; the takeout is skimmed off the top and the remainder is distributed proportionately to the winning ticketholders. That is, given X and specified payoff rate of $P^*>0$, to be skimmed off the top, W is defined as follows:

$$w_i = \frac{1-P^*}{x_i} - 1, \quad i = 1,2,\ldots,n.$$

This way of defining W makes the given X the maximin hedge for W that yields the profit rate P^*. For from the definition of w_i we obtain $x_i = (1-P^*)/(1+w_i)$ and $x_i/(1-P^*) = (1+w_i)^{-1}$, which gives

$$\sum_{i=1}^{n} \frac{1}{1+w_i} = \frac{\sum_{i=1}^{n} x_i}{1-P^*} = \frac{1}{1-P^*}.$$

Thus, putting j for i, we have by this definition

$$x_j = \frac{1}{\frac{1}{x_j}} = \frac{1}{\frac{1-P^*}{x_j}\cdot\frac{1}{1-P^*}} = \frac{1}{(1+w_j)\sum_{i=1}^{n}\frac{1}{1+w_i}} = x_j^*$$

as found earlier.

Can a client bettor beat the house at its own game? In other words, can a bettor hedge his bets in parimutuel betting or against a bookmaker operating at or near his (positive) maximin in such a way as to guarantee a sure profit? This is one version of a question gamblers loosely refer to as "dutching". It should surprise no one that the answer is generally no. Insight into that fact can be gotten by comparing FIGURE (1a) and FIGURE (1b). To be more precise, given W, if a client bettor spreads his money on B_i in the ratio of $x_i:1$, his payoff rate is given by

$$\tilde{P}(X,Y) = -P(X,Y).$$

By an argument parallel to that given from the viewpoint of the house, we can verify that the client's maximin hedge is the same as the house's and that it yields a payoff rate of $-P^*$.

Often the question of "dutching" arises when a bettor has "inside information" that a subset of outcomes of the experiment will *not* happen (e.g., he knows one or more horses that will *not* win the race). In that case a maximin hedge may yield a positive (sure) profit. Suppose there is a subset of $I \subset \{1,2,\ldots,n\}$ such that $\{B_i: i\in I\}$ exhaust all the "real" possibilities. Then the maximin hedge defined by

$$x_i = \begin{cases} 0, & \text{if } i \notin I, \\ \dfrac{1}{(1+w_i)\sum_{j\in I}\frac{1}{1+w_j}} & \text{if } i \in I, \end{cases}$$

yields a positive payoff rate if and only if

$$\sum_{i\in I}\frac{1}{1+w_i} < 1.$$

Note that in parimutuel betting, W can only be estimated before the race and that it is affected by the money put up by the player. Horse betting under general conditions is discussed in **[2]**.

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PROBLEMS

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Proposals

To be considered for publication, solutions should be mailed before December 1, 1978.

1039. Sum the series $\sum_{k=1}^{\infty} \frac{1}{k^2} \tan\frac{k\pi}{m} \tan\frac{k\pi}{n}$. [*M. B. Gregory & J. M. Metzger, University of North Dakota.*]

1040. If A is an $m\times n$ matrix which is not invertible, show that there are infinitely many $n\times m$ matrices X satisfying $AXA=A$. [*H. Kestelman, University College, London.*]

1041. For $0<m<n$, find $N(m,n)$, the minimum positive integer such that any subset of $\{1,2,\ldots,n\}$ of $N(m,n)$ elements contains numbers differing by m. [*Richard A. Gibbs, Fort Lewis College.*]

1042. Prove that any integer which is the sum of the squares of two different, non-zero integers is divisible by a prime which is the sum of the squares of two different, non-zero integers. [*Henry Klostergaard, California State University, Northridge.*]

1043. If (a_i,b_i,c_i) are the sides, R_i the circumradii, r_i the inradii, and s_i the semi-perimeters of two triangles $(i=1,2)$, show that

$$\left\{\frac{s_1}{r_1R_1}\frac{s_2}{r_2R_2}\right\}^{1/2} \geqslant 3\left\{\frac{1}{\sqrt{a_1a_2}}+\frac{1}{\sqrt{b_1b_2}}+\frac{1}{\sqrt{c_1c_2}}\right\} \tag{1}$$

with equality iff the two triangles are equilateral.

Also show that the analogous three triangle inequality

$$\left\{\frac{s_1}{r_1R_1}\frac{s_2}{r_2R_2}\frac{s_3}{r_3R_3}\right\}^{1/2} \geqslant 9\left\{\frac{1}{\sqrt{a_1a_2a_3}}+\frac{1}{\sqrt{b_1b_2b_3}}+\frac{1}{\sqrt{c_1c_2c_3}}\right\} \tag{2}$$

is invalid. [*M. S. Klamkin, University of Alberta.*]

ASSISTANT EDITORS: DON BONAR, *Denison University*; WILLIAM A. MCWORTER, JR., *The Ohio State University. We invite readers to submit problems believed to be new. Proposals should be accompanied by solutions, when available, and by any information that will assist the editors. Solutions to published problems should be submitted on separate, signed sheets. An asterisk (*) will be placed by a problem to indicate that the proposer did not supply a solution. A problem submitted as a Quickie should be one that has an unexpected succinct solution. Readers desiring acknowledgement of their communications should include a self-addressed stamped card. Send all communications to this department to Dan Eustice, The Ohio State University, 231 W. 18th Ave., Columbus, Ohio 43210.*

1044. Let p be a prime and k a positive integer. The congruence relation $(x-a)(x-b)\equiv 0 \pmod{p^k}$ has the obvious solutions $x\equiv a \pmod{p^k}$ and $x\equiv b \pmod{p^k}$. When are these the only solutions? [*J. Metzger, University of North Dakota.*]

1045. N is a perfect square if there exists a K such that $N=K^2$. Following T. N. Bhargava and P. H. Doyle, *On the Existence of Absolute Primes*, (47) 233–234, this MAGAZINE, define N to be an *absolute perfect square*, relative to a given base, if every permutation of the digits of N is a perfect square in that base. In base ten, 1, 4, and 9 are obviously absolute perfect squares. Show that these are the only ones. [*J. L. Murphy, California State College at San Bernardino.*]

1046. For an arbitrary positive integer k, consider the decimal integer h consisting of m copies of k followed by n zeros. Show that for each positive integer x, there exist an m, $m\neq 0$, and an n such that x divides h. [*Daniel J. Aulicino, Fiorello H. LaGuardia Community College.*]

1047. Given an infinite sequence $A=\{a_n\}$ of positive integers, we define a family of sequences A_i, where $A_0=A$ and $A_i=\{b_r\}$ for $i=1,2,3,\ldots$, where b_r is the number of times that the rth lowest term of A_{i-1} occurs in A_{i-1}. For example, if $A=A_0=\{1,2,2,3,3,3,4,4,4,4\ldots.\}$, then $A_1=\{1,2,3,4,\ldots.\}$ and $A_2=\{1,1,1,1,\ldots.\}$.

(a) Find a non-decreasing sequence A such that the sequences A_i are all distinct.

(b)* Let $T=\{t_n\}$ be the unique non-decreasing sequence containing all the positive integers which has the property that $T_1=T_0$. Define $U=\{u_n\}$ and $V=\{v_n\}$ so that for all n, $u_n=t_{2n-1}$ and $v_n=t_{2n}$. Are the sequences U_i and V_i all distinct? [*James Propp, Great Neck, New York.*]

Quickies

Solutions to Quickies appear at the conclusion of the Problems section.

Q653. Show that if $(a,b)=1$, then the set of integers $\{ai:0\leqslant i\leqslant r-1\}\cup\{bj:1\leqslant j\leqslant s\}$, where $r+s=a+b$, forms a complete set of residues mod $(a+b)$. [*L. Kuipers, Mollens, Switzerland.*]

Q654. Find all acute triangles with integral sides and a 60° angle in which the sides adjacent to the 60° angle differ by unity. [*George Berzsenyi, Lamar University.*]

Solutions

Orthogonal Basis **May 1976**

984. Let $(b_1,b_2,\ldots,b_n)$ be a non-zero element of $\mathbf{R}^n$. For which n, $2\leqslant n\leqslant 8$, is it true that one can choose an orthogonal basis for $\mathbf{R}^n$ from the collection

$$\{(\pm b_{\pi(1)},\pm b_{\pi(2)},\ldots,\pm b_{\pi(n)}):\pi\in P_n\},$$

where P_n is the set of all permutations of $(1,2,\ldots,n)$? [*Peter Ørno, The Ohio State University.*]

Solution: 1) The permissible values of n are 2, 4, and 8 as follows:

$$\begin{bmatrix} b_1 & b_2 \\ b_2 & -b_1 \end{bmatrix}, \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ b_2 & -b_1 & b_4 & -b_3 \\ b_3 & -b_4 & -b_1 & b_2 \\ b_4 & b_3 & -b_2 & -b_1 \end{bmatrix}, \begin{bmatrix} b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & b_8 \\ b_2 & -b_1 & b_4 & -b_3 & b_6 & -b_5 & b_8 & -b_7 \\ b_3 & -b_4 & -b_1 & b_2 & -b_7 & b_8 & b_5 & -b_6 \\ b_4 & b_3 & -b_2 & -b_1 & b_8 & b_7 & -b_6 & -b_5 \\ b_5 & -b_6 & b_7 & -b_3 & -b_1 & b_2 & -b_3 & b_4 \\ b_6 & b_5 & -b_8 & -b_7 & -b_2 & -b_1 & b_4 & b_3 \\ b_7 & -b_8 & -b_5 & b_6 & b_3 & -b_4 & -b_1 & b_2 \\ b_8 & b_7 & b_6 & b_5 & -b_4 & -b_3 & -b_2 & -b_1 \end{bmatrix}.$$

2) The b_i's may be considered to be variables. So for an inner product to vanish, it follows that $\pi(i)=j$ implies $\pi(j)=i$, and $\pi(i)\neq i$. This shows n must be even. Also, $\pi(j)=i$ for exactly one π. (One vector may be chosen to be $[b_1b_2b_3b_4\cdots b_n]$, and this was done above.)
3) $n=6$ does not work. Without loss of generality, the first two vectors may be chosen

$$\begin{bmatrix} b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ b_2 & -b_1 & b_4 & -b_3 & b_6 & -b_5 \end{bmatrix}$$

By part 2) one other vector has b_3 as its first component. This forces the vector to be

$$\begin{bmatrix} b_3 & -b_4 & -b_1 & b_2 & x & x \end{bmatrix}$$

where the x's must be $\pm b_5$ or $\pm b_6$. This contradicts the last part of 2).

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Remark: Square matrices of letters which commute in pairs, and to which signs are attached so that the inner product of any two distinct row vectors is zero have been studied extensively. Such a matrix is called a *Hadamard design* (on n letters). In the paper "Hadamard design" by Jennifer Wallis (Bull. Austral. Math Soc. Vol. 2 (1970), 45–54), it is proved that a Hadamard design on n letters exists if and only if $n=2$, 4, or 8. This gives a complete answer to our present problem.

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Editors' Comment. Every solver solved the intended formulation of the problem: "For which n is it true that for *every* non-zero vector b in R^n one can find $n-1$ signed permutations of b so that the n vectors form an orthogonal basis for R^n." A discussion of the related problem involving the existence of division algebras can be found in the article by Charles W. Curtis in Studies in Modern Algebra. (Studies in Mathematics, Vol 2., Mathematical Association of America.)

However, for *a* specific b a solution may exist for other n. For every n, the standard basis vector $(1,0,\ldots,0)$ and its $n-1$ permutations form an orthogonal basis for R^n. If we require *all* components to be non-zero, there still may be individual vectors b in R^n which, together with $n-1$ of their signed permutations, form an orthogonal basis. In particular, with $n=3$ and $b_1=b_2b_3/(b_2+b_3)$, we have the orthogonal basis (b_1,b_2,b_3), $(b_2,-b_3,b_1)$, $(b_3,b_1,-b_2)$. One such b is a favorite of courses on $R^3:(1,2,2)$.

Also solved by the proposer.

The Best Constant? **May 1976**

986. Show that there exists a constant c such that $a+b<n+c\ln n$, for all positive integers a, b, and n for which $n!/(a!b!)$ is an integer. [*Paul Erdös, Hungarian Academy of Science.*]

Solution: For each positive integer n, define $t(n)$ to be the exponent of the highest power of 2 which divides n, and let $d=\log_2 n$. Now it is easily shown that

$$t(n!)=\sum_{k=1}^{\infty}\left[\frac{n}{2^k}\right]=\sum_{1\leqslant k\leqslant d}\left[\frac{n}{2^k}\right],$$

where the brackets denote the greatest integer function. Thus, if $a!b!$ divides $n!$, then

$$t(a!b!)=t(a!)+t(b!)=\sum_{k=1}^{\infty}\left[\frac{a}{2^k}\right]+\left[\frac{b}{2^k}\right]\geqslant\sum_{1\leqslant k\leqslant d}\left[\frac{a-2^k}{2^k}+\frac{b-2^k}{2^k}\right]$$

$$=(a+b)\sum_{1\leqslant k\leqslant d}2^{-k}-2[d]=(a+b)B-2[d],$$

where $B=\sum_{k=1}^{[d]}2^{-k}=1-2^{-[d]}$, and

$$t(n!)=\sum_{k=1}^{\infty}\left[\frac{n}{2^k}\right]\leqslant n\sum_{k=1}^{[d]}2^{-k}=Bn.$$

Now, since $t(a!b!)\leqslant t(n!)$,

$$a+b\leqslant\frac{t(a!b!)+2[d]}{B}\leqslant\frac{t(n!)}{B}+\frac{2[d]}{B}\leqslant n+\frac{2}{B}\cdot\frac{\ln n}{\ln 2}$$

$$\leqslant n+\frac{8}{3\ln 2}\ln n<n+4\ln n$$

if $n\geqslant 4$.

Adam Riese
Wright State University

Also solved by Eli Leon Isaacson, Jordan I. Levy, and the proposer.

Thumbtacks **November 1976**

996. Suppose thumbtacks are used to tack congruent square sheets of paper to a large bulletin board subject to the following conditions:

(i) The sides of the sheet are parallel to the sides of the bound;
(ii) Each sheet has exactly four thumbtacks, one in each corner; and
(iii) The sheets may overlap slightly so that one thumbtack could secure a corner of from one to four sheets.

(a) Find, in terms of n, the minimum number of thumbtacks required to tack n such sheets.
(b)* For a given n, find the number of distinct minimal arrangements.
(c)* Can the problem be generalized to hypercubes and hyperthumbtacks in three or more dimensions? [*Richard A. Gibbs, Fort Lewis College.*]

Solution: (a) Let $g(n)$ denote the required minimum. Define p to be the unique integer for which $p^2\leqslant n<(p+1)^2$, and let $n=p^2+q$ $(0\leqslant q\leqslant 2p)$. It will be shown that

$$g(n)=\begin{cases}(p+1)^2 & \text{if } q=0\\(p+1)^2+q+1 & \text{if } 1\leqslant q\leqslant p\\(p+1)^2+q+2 & \text{if } p+1\leqslant q\leqslant 2p\end{cases}\qquad(*)$$

Suppose n sheets are on the board in a minimal configuration (i.e., $g(n)$ tacks are used). It will first be shown that the tacks all lie on a square lattice in which the spacing of points is the width of a sheet.

Define two sheets to be equivalent if their corners lie on the same square lattice. This defines an equivalence relation among the sheets. It must be shown there is only one equivalence class. Suppose there are at least two. Clearly, sheets in two different equivalence classes cannot share a tack. Therefore, moving all sheets of one equivalence class rigidly to the right of all the other sheets does not affect the number of tacks needed. But now the equivalence class at the right may be moved rigidly to the left (as well as up and down) so that one left-most sheet in the class will share two tacks with one right-most sheet not in the class. This shows the original configuration was not minimal.

It may therefore be assumed that the sheets lie between the horizontal and vertical lines of a grid whose spacing is the width of a sheet and which includes the top and left side of the board. Number the horizontal lines consecutively, starting from zero at the top of the board. Assume the sheets lie above horizontal number k. Let t_j be the number of tacks on horizontal j, and let r_j be the number of sheets between horizontals $j-1$ and j. Clearly, $g(n)=t_0+\cdots+t_k$, and $n=r_1+\cdots+r_k$. Also, it is easily shown by induction that the minimal number of tacks required to tack down the tops (or bottoms) of r sheets which lie between two consecutive horizontals is $r+1$. (The minimum is assumed if the sheets have no gaps between them.) Therefore, $t_0 \geqslant r_1+1$, $t_k \geqslant r_k+1$, and $t_j \geqslant \max(r_j, r_{j+1})+1$ for $1 \leqslant j \leqslant k$.

Make a new configuration by moving the sheets to the left as far as possible to fill in all of the (horizontal) gaps. Let s_j be the new number of tacks on horizontal j. Since there are no gaps in any row, it follows that $s_0=r_1+1$, $s_k=r_k+1$, and $s_j=\max(r_j, r_{j+1})+1$ for $1 \leqslant j \leqslant k$. So the number of tacks is not increased. Since the original configuration is minimal, $t_j=s_j$ for $0 \leqslant j \leqslant k$.

Similarly, by moving the sheets up as far as possible to remove all of the (vertical) gaps (after the sheets have been moved to the left), the number of tacks used remains minimal. However, in this new configuration, $r_1 \geqslant r_2 \geqslant \cdots \geqslant r_k$ where r_j is defined as above but for the new configuration. (Define the t_j's similarly.) Then

$$\begin{aligned} g(n) = t_0+\cdots+t_k &= (r_1+1)+(\max(r_1,r_2)+1)+\cdots+ \\ &\quad +(\max(r_{k-1},r_k)+1)+(r_k+1) \\ &= (r_1+1)+(r_1+1)+(r_2+1)+\cdots+(r_{k-1}+1)+(r_k+1) \\ &= r_1+(r_1+r_2+\cdots+r_k)+(k+1)=r_1+n+k+1. \end{aligned}$$

Since all the sheets are contained in r_1 columns and k rows, it follows that $r_1 k \geqslant n$. Therefore, the following lemma shows that (*) gives a lower bound for $g(n)$.

LEMMA. *If* $n=p^2+q$ $(0 \leqslant q \leqslant 2p)$, *then*

$$\min(r+k : rk \geqslant n,\ r \text{ and } k \text{ pos. int.}) = \begin{cases} 2p & q=0 \\ 2p+1 & 1 \leqslant q \leqslant p \\ 2p+2 & p+1 \leqslant q \leqslant 2p \end{cases} \qquad (**)$$

To see that (*) gives an upper bound, start with $n=p^2$ and arrange the sheets in a square. This requires $(p+1)^2$ tacks. Successively add sheets down a side $(1 \leqslant q \leqslant p)$ and then across the bottom $(p+1 \leqslant q \leqslant 2p)$. This shows (*) is actually attained.

All that is left is:

Proof of Lemma: The case $n=p^2$ is trivial, so assume $p^2<n<(p+1)^2$ $(1 \leqslant q \leqslant 2p)$. Taking $r=p$, $k=p+1$ for $1 \leqslant q \leqslant p$ and $r=p+1$, $k=p+1$ for $p+1 \leqslant q \leqslant 2p$ shows that (**) gives an upper bound for the minimum.

To see that it gives a lower bound, note that $r+k \geqslant 2\sqrt{rk} \geqslant 2\sqrt{n} > [2\sqrt{n}\,]$. Hence, $r+k \geqslant [2\sqrt{n}\,]+1$. Writing $n=p^2+x$ for $0<x<2p+1$ shows that

$$\begin{aligned} 2p<2\sqrt{n}<2p+1 \quad &\text{for} \quad 0<x<p+1/4 \\ 2p+1<2\sqrt{n}<2p+2 \quad &\text{for} \quad p+1/4<x<2p+1. \end{aligned}$$

Therefore, we obtain in the lower bound

$$r+k \geq [2\sqrt{n}\,]+1=\begin{cases} 2p+1 & \text{for} \quad 1 \leq q \leq p \\ 2p+2 & \text{for} \quad p+1 \leq q \leq 2p \end{cases}.$$

This gives the lower bound.

ELI L. ISAACSON
New York University

Part (a) was also solved by the proposer. Partial solutions to part (a) were obtained by Thomas E. Elsner, Donald C. Fuller, Michael Goldberg, and J. Rue. No solutions to parts (b) and (c) were submitted.

Lott's Problem **November 1976**

997. Let P be a polynomial of degree n, $n \geq 2$, with simple zeros $z_1, z_2, \ldots, z_n$. Let $\{g_k\}$ be the sequence of functions defined by $g_1 = 1/P'$, and $g_{k+1} = g_k'/P'$. Prove for all k that $\sum_{j=1}^{n} g_k(z_j) = 0$. [*John Lott, Student, Southwest High School, Kansas City, Missouri.*]

Solution I: By induction. For $k=1$, let $P(z)=(z-z_1)\ldots(z-z_n) \equiv Q(z)(z-z_n)$. Then, $P'(z)=Q(z) + Q'(z)(z-z_n)$. Thus $P'(z_j)=Q'(z_j)(z_j-z_n)$ for $1 \leq j \leq n-1$ and $P'(z_n)=Q(z_n)$. By partial fractions and then setting $z=z_n$,

$$\frac{1}{Q(z)}=\sum_{j=1}^{n-1} \frac{1/Q'(z_j)}{z-z_j}=\sum_{j=1}^{n-1} \frac{z_j-z_n}{P'(z_j)(z-z_j)}$$

$$\frac{1}{Q(z_n)}=\frac{1}{P'(z_n)}=\sum_{j=1}^{n-1} \frac{-1}{P'(z_j)} \quad \text{or} \quad \sum_{j=1}^{n} \frac{1}{P'(z_j)}=0.$$

Now, assume that $\sum g_k(z_j)=0$ for some k and all polynomials P with simple zeros, and let $R(z)=P(z)+c$ for some constant c. For $|c|$ sufficiently small, R has simple zeros, $w_j=w_j(c)$, $1 \leq j \leq n$. Also, since $P'=R'$, $g_k(z)=g_k(z;P)=g_k(z;R)$. From the Taylor series for P we have $P(z)=P'(z_j)(z-z_j)+O((z-z_j)^2)$ so that

$$-c=P(w_j)=P'(z_j)(w_j-z_j)+O\big((w_j-z_j)^2\big) \qquad \text{and} \qquad w_j-z_j \sim \frac{-c}{P'(z_j)} \quad \text{as} \quad c \to 0.$$

Since R is also a polynomial with simple zeros, our indication assumption implies both $\sum_{j=1}^{n} g_k(z_j)=0$ and $\sum_{j=1}^{n} g_k(w_j)=0$. Thus

$$0=\sum_{j=1}^{n} g_k(w_j)-\sum_{j=1}^{n} g_k(z_j)=\sum_{j=1}^{n} g_k'(z_j)(w_j-z_j)+O\big((w_j-z_j)^2\big)$$
$$=-\sum_{j=1}^{n} \frac{g_k'(z_j)c}{P'(z_j)}+O(c^2)$$

That is,

$$\sum_{j=1}^{n} \frac{g_k'(z_j)}{P'(z_j)} \equiv \sum_{j=1}^{n} g_{k+1}(z_j)=O(c),$$

which completes the induction.

JOHN LOTT
Kansas City, Missouri

Solution II: For each j, $1 \leqslant j \leqslant n$, let D_j be a closed disk about z_j as center in which P is univalent. For each k define $G_{j,k}(z)$ in the interior of D_j by

$$G_{j,k}(z) = \frac{(k-1)!}{2\pi i} \oint_{C_j} \frac{d\zeta}{(P(\zeta) - P(z))^k}, \tag{1}$$

where C_j is the boundary circle of D_j. Then, for $k=1$,

$$G_{j,1}(z) = \frac{1}{2\pi i} \oint_{C_j} \frac{d\zeta}{P(\zeta) - P(z)} = \frac{1}{P'(z)}$$

since, as a function of ζ, $1/(P(\zeta) - P(z))$ has a simple pole at z with residue $1/P'(z)$. Also, we find that

$$G'_{j,k}(z) = \left(\frac{k!}{2\pi i} \oint_{C_j} \frac{d\zeta}{(P(\zeta) - P(z))^{k+1}} \right) \cdot P'(z).$$

Thus, $G_{j,k+1}(z) = G'_{j,k}(z)/P'(z)$ and therefore $G_{j,k}(z)$ agrees with $g_k(z)$ in D_j, and consequently

$$g_k(z_j) = \frac{(k-1)!}{2\pi i} \oint_{C_j} \frac{d\zeta}{(P(\zeta))^k}. \tag{2}$$

Choosing R so large that the circle C_R with center 0 and radius R contains in its interior all the z_j's and applying the residue theorem we have, for $k=1,2,\ldots$

$$\sum_{j=1}^{n} g_k(z_j) = \frac{(k-1)!}{2\pi i} \oint_{C_R} \frac{d\zeta}{(P(\zeta))^k} = 0$$

since $\deg P^k \geqslant 2$ for all k.

PETER ØRNO
The Ohio State University

Editors' Comment. M. S. Klamkin gives an extension by starting with the partial fraction expansion

$$\frac{1}{(P(z))^m} = \sum_{i=1}^{m} \left\{ \frac{A_{i1}}{z - z_i} + \frac{A_{i2}}{(z - z_i)^2} + \cdots + \frac{A_{im}}{(z - z_i)^m} \right\}$$

where the A_{ij} are functions of the roots. By expanding both sides of this equation in powers of $1/z$ and equating the coefficients of like powers of z, he obtains the following identities:

$$\sum_i A_{i1} = 0, \quad \sum_i (A_{i1} z_i + A_{i2}) = 0, \quad \sum_i (A_{i1} z_i^2 + 2A_{i2} z_i + A_{i3}) = 0, \text{ etc.}$$

He then shows $(m-1)! A_{i1} = g_m(z_i)$ and derives other recurrence relations for the remaining A_{ij}.

Also solved by J. M. Stark.

A 120° Triangle **November 1976**

998. Characterize all triangles in which the triangle whose vertices are the feet of the internal angle bisectors is a right triangle. [*Hüseyin Demir, Middle East Technical University, Ankara, Turkey.*]

Solution: Let A', B', C' be the feet of the angle bisectors of angles A, B, C, respectively. Then angle $A'C'B'$ is a right angle iff angle ACB is 120 degrees.

Let a,b,c (a',b',c') be the lengths of sides opposite A,B,C (A',B',C'), respectively. Using the law of cosines and the fact that the angle bisector divides the opposite side in the ratio of the adjacent sides it follows that:

$$(c')^2=\left(\frac{ab}{a+c}\right)^2+\left(\frac{ab}{b+c}\right)^2-2\left(\frac{ab}{a+c}\right)\left(\frac{ab}{b+c}\right)\left(\frac{a^2+b^2-c^2}{2ab}\right)$$

$$(b')^2=\left(\frac{ac}{b+c}\right)^2+\left(\frac{ac}{a+b}\right)^2-2\left(\frac{ac}{b+c}\right)\left(\frac{ac}{a+b}\right)\left(\frac{a^2+c^2-b^2}{2ac}\right)$$

$$(a')^2=\left(\frac{bc}{a+c}\right)^2+\left(\frac{bc}{a+b}\right)^2-2\left(\frac{bc}{a+c}\right)\left(\frac{bc}{a+b}\right)\left(\frac{b^2+c^2-a^2}{2bc}\right)$$

Angle $A'C'B'$ is a right angle iff $(a')^2+(b')^2-(c')^2=0$. But this equation simplifies (after much algebra) to

$$\frac{2abc^2(a^2+b^2-c^2+ab)}{(a+b)^2(a+c)(b+c)}=0.$$

Thus angle $A'C'B'$ is a right angle iff $a^2+b^2-c^2+ab=0$. But the law of cosines yields $a^2+b^2-c^2+ab=0$ iff angle ACB is 120°.

John Oman
University of Wisconsin-Oshkosh

Also solved by Gordon Bennett, Howard Eves, Michael Goldberg, Leonard D. Goldstone, M. G. Greening (Australia), Hubert J. Ludwig, J. M. Stark, Pambuccian Victor (Romania), Robert L. Young, and the proposer.

Factorial Factors **November 1976**

999. Let $\{a_i\}$ and $\{b_i\}$, $i=1,2,\ldots,k$, be natural numbers arranged in non-decreasing order. For which values of k is it true that $\Sigma_{i=1}^{k}(a_i!)=\Sigma_{i=1}^{k}(b_i!)$ implies $a_i=b_i$ for all i? What is the corresponding result if the two sequences are strictly increasing? [*Joseph Silverman, Brown University.*]

Solution: The implication is not valid for $k=5$ since $2!+2!+2!+2!+2!=1!+1!+1!+1!+3!$. And it is not valid for $k>5$, since $k-5$ identical terms can be added to both sides of this last equality. However if $k=1$, 2, 3 or 4, the implication is correct. Certainly if $a_1!=b_1!$ then $a_1=b_1$, and for $k>1$ either $a_k=b_k$, in which case these two terms may be dropped from the sums producing equality with $k-1$ terms on both sides, or $a_k\neq b_k$. For the latter possibility it can be assumed without loss of generality that $a_k<b_k$. Then $b_k!\geqslant(a_k+1)!$ and $ka_k!\geqslant\Sigma(a_i!)=\Sigma(b_i!)>b_k!\geqslant(a_k+1)a_k!$, i.e. $k-1>a_k$. This inequality simplifies the verification for $k\leqslant 4$ that $a_k\neq b_k$ cannot occur.

If the two series were strictly increasing, the implication would be valid for all k, since then $a_k\geqslant k$, which would contradict $k-1>a_k$ were $a_k<b_k$.

Graham Lord
Université Laval
Québec, Canada

Also solved by Thomas E. Elsner, Donald C. Fuller, Marguerite F. Gerstell, Eli L. Isaacson, Jordan I. Levy, Pambuccian Victor (Romania), Edward T. H. Wang (Canada), Paul Y. H. Yiu (Hong Kong), and the proposer.

1000. T denotes a cyclic permutation operator acting on the indices of a sequence $\{a_i\}$, that is, $T(a_1x_1+a_2x_2+\cdots+a_nx_n)=a_2x_1+a_3x_2+\cdots+a_1x_n$. If, for all i, $a_i \geqslant 0$ and $x_i > 0$, show that

$$\left\{\sum_{i=1}^{n}\frac{a_i}{n}\right\}^n \geqslant \prod_{i=1}^{n} T^i\left\{\frac{a_1x_1+a_2x_2+\cdots+a_nx_n}{x_1+x_2+\cdots+x_n}\right\} \geqslant \prod_{i=1}^{n} a_i.$$

[*Murray S. Klamkin, University of Alberta.*]

Solution: Let

$$w_i = T^i\left\{\frac{a_1x_1+\cdots+a_nx_n}{x_1+\cdots+x_n}\right\}.$$

Since the geometric mean of the w_i's never exceeds their arithmetic mean, we have

$$\sum_{i=1}^{n}\frac{a_i}{n}=\sum_{i=1}^{n}\frac{w_i}{n}\geqslant\left(\prod_{i=1}^{n} w_i\right)^{1/n}$$

which yields the left-hand inequality. Also, from the arithmetic mean-geometric mean inequality,

$$T^i\left\{\frac{\Sigma a_ix_i}{\Sigma x_i}\right\}\geqslant\left\{a_i^{x_1}\cdots a_n^{x_{n-i+1}}a_1^{x_{n-i+2}}\cdots a_{i-1}^{x_n}\right\}^{1/\Sigma x_i}$$

so that

$$\prod_{i=1}^{n} T^i\left\{\frac{\Sigma a_ix_i}{\Sigma x_i}\right\}\geqslant\prod_{i=1}^{n} a_i.$$

JERRY METZGER
University of North Dakota

Also solved by Jesse Deutsch, Donald C. Fuller, Ralph Garfield, M. G. Greening (Australia), Richard A. Groeneveld, Eli L. Isaacson, Mark Kleiman, Kenneth Klinger, and the proposer.

Answers

Solutions to the Quickies which appear near the beginning of the Problems section.

Q653. Since $(a,b)=1$, we have $(a,a+b)=1$. Hence the set $\{ai: 0\leqslant i\leqslant a+b-1\}$ forms a complete set of residues mod$(a+b)$. However, $a(a+b-j)\equiv jb(\bmod a+b)$ for $j=1,2,\ldots,s$, yielding result. (This is an extension of a lemma of Eckford Cohen, *The distribution of certain sequences of integers*, Amer. Math. Monthly, 70 (1963) 516–521.)

Q654. There are no such triangles. By the law of cosines, the Diophantine equation under consideration is $x^2+(x+1)^2-x(x+1)=z^2$. But one may write this as $x=(x+1-z)(x+1+z)$, which clearly has no solutions in positive integers. (One can show that the same problem with a 120° angle has infinitely many solutions.)

REVIEWS

Paul J. Campbell, Editor
Beloit College

Pierre Malraison, Editor
Control Data Corp.

Articles and books are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of the mathematics literature. Some reviews of books are adapted from the Telegraphic Reviews in the American Mathematical Monthly.

Kolata, Gina Bari, *Cryptology: a secret meeting at the IDA?*, Science 200 (14 April 1978) 184.

Mathematics and politics don't often overlap sufficiently for intrigues and secret meetings to take place. Here is (maybe) an example of such an overlap. A meeting at the Institute for Defense Analysis called to develop software for the Institute's new Cray computer may have been a meeting on cryptology. The underlying mathematical problem involved in breaking a new class of codes is factoring large numbers into primes; existing algorithms take too long and this meeting may have been a discussion of possible improvements in existing algorithms.

Smullyan, Raymond M., What is the Name of This Book? The Riddle of Dracula and Other Logical Puzzles, P-H, 1978; 241 pp, $8.95.

An extraordinary collection of logical puzzles and anecdotes, ranging from silly old chestnuts to intricate concatenations of liar enigmas all engagingly expressed by a cast of history's most illogical characters. Concludes with a series of puzzles that express the essence of Gödel's Incompleteness Theorem. A full prepublication review appeared in Martin Gardner's March 1978 column in *Scientific American.*

Stewart, Ian, *What shape is a catastrophe?*, Analog (June 1978) 29-49.

No science fiction stories have appeared yet about catastrophe theory; this is a "science fact" article by one of the practitioners, appearing in a national science fiction magazine. After a discussion of the catastrophe theory controversy, Stewart discusses several uncontroversial applications to optics, physics and ship design. The article concludes with a look at applications to the "soft" sciences and a catastrophe model of the catastrophe theory controversy. "The critical wolves who bare their teeth solely at [the] most speculative end of CT are for all their huffing and puffing, blowing down only houses of straw. But the little pigs of CT can build brick houses too."

Steen, Lynn Arthur, *What's in a game?*, Science News 113 (1 April 1978) 204-206.

An account of the growing interaction between mathematical theories and games of strategy. "Both the mathematician and the game player rely on extensive hypothetical ('what if...') reasoning to develop sound strategy." Eight games are described, with rules given and gameboards depicted.

Eves, Howard W., Mathematical Circles Adieu, Prindle, 1977; xxiii + 181 pp, $10.

An unintended sequel to the author's four well-known previous *Circles*, motivated and fueled by readers' contributions.

Saaty, Thomas L. and Kainen, Paul C., The Four Color Problem: Assaults and Conquest, McGraw-Hill, 1977; ix + 217 pp.

"A funny thing happened to this book on the way to the publisher"--a manuscript had already been completed when it was announced in 1976 that the 4CC had been proved. The authors responded by revising the book to include a complete account of the methodology: how the solution evolved, plus the relevant graph theory needed to understand it. The rest of the book is devoted to expounding equivalents of the 4CC, based on Saaty's award-winning "Thirteen colorful variations on Guthrie's four-color conjecture," *Amer. Math. Monthly* 79 (1972) 2-43. This excellent book is flawed only in the comprehensive bibliography, which deserved better proofreading and some updating.

Graham, R.L., *The combinatorial mathematics of scheduling*, Scientific American 238 (March 1978) 124-132, 154.

"Apparently logical ways of constructing schedules cannot be counted on to perform equally well in different situations...in some instances increasing the number of workers on a job can actually increase the time required to meet the schedule. Some of the commonest and most intuitive scheduling procedures can give rise to unexpected and even seemingly paradoxical results." The author illustrates these assertions and acquaints the reader with the current state of the art of scheduling.

Kolata, Gina Bari, *A new, rapid method for Fourier transforms*, Science 199 (24 March 1978) 1326-1327.

Shmuel Winograd (IBM) has found a new way to calculate discrete Fourier transforms, that is, to separate a signal into its component frequencies. His method requires the same number of additions, 80% fewer multiplications, but a little more storage and movement of data than the Fast Fourier Transform (FFT) of Cooley and Tukey. The new result is significant because the FFT is one of the most frequently used mathematical techniques, having revolutionized data analysis in molecular spectroscopy, seismology and signal processing for speech analysis, radar, and sonar.

Schattschneider, Doris and Walker, Wallace, M.C. Escher Kaleidocycles, Ballantine, 1977; iv + 43 pp + 17 multicolor cardboard punch-out models, $8.95 (P).

Simple folding of a triangular grid can produce a surprising three-dimensional result: a ring of tetrahedra that can be rotated through its center, bringing different faces to view. The authors call such models Kaleidocycles. Their splendid idea of decorating the surfaces with Escher prints brought into play technical, mathematical and artistic considerations: parallel projection of the patterns to fit onto the solids, adjusting some coloring patterns to ensure perfect matching at each rotation, and hand-painting of the colors. Assembling the punch-out models allows the reader to participate in creation of these marvelous forms; the folding and gluing requires patience and precision, so younger children may need help. The results are a joy to hold, behold, and tumble through themselves!

Goldstine, Herman H., A History of Numerical Analysis from the 16th Through the 19th Century, Springer-Verlag, 1977; xiv + 348 pp, $24.80.

A faithful modern exposition of major documents by the masters (especially Napier and Briggs, Newton, Euler, Lagrange, Laplace, Gauss, Jacobi, Cauchy) on logarithms, interpolation, summation, least squares, quadrature and related themes. Not a comprehensive "search-for-origins" history, but a monument to the author's feeling that the great masters were those responsible for the most significant accomplishments. An invaluable sourcebook in the classics of numerical analysis.

Wolbarst, A.B., *An intuitive approach to group representation theory*, American J. Physics 45 (September 1977) 803-810.

An exposition of irreducible representations through the use of elementary real-space (as opposed to function-space) examples.

Halberstam, H., *Some unsolved problems in higher arithmetic*, in Duncan, Ronald and Weston-Smith, Miranda (Eds.), The Encyclopedia of Ignorance, Pergamon Pr, 1977; $30, $15 (P), pp. 191-204.

A potpourri of unsolved problems in elementary and analytic number theory, from Goldbach's conjecture to the Riemann hypothesis.

Penrose, Roger, *Is nature complex?*, in Duncan, Ronald and Weston-Smith, Miranda (Eds.), The Encyclopedia of Ignorance, Pergamon Pr, 1977; $30, $15 (P), pp. 160-166.

Application of complex numbers and their geometry to the use of twistors (elements of C^4) in quantum mechanics.

Jaworski, John; Mason, John; Slomson, Alan, et al., Chez Angelique: The Bumper Late-Night Problem Book, available from John Mason, Faculty of Mathematics, The Open University, Walton Hall, Milton Keynes, MK7 6AA, England; 1975; 64 pp, £1.50 (P).

Chez Angelique "was simply a mathematical nightclub--a cross between a cabaret-act and a folk-club perhaps" which took place at the Open University's residential Summer School at Stirling University. "Tutors and students alike got up from the floor to present their puzzles. The range was enormous...Best of all was the enthusiasm...People fought to get in..." An extraordinary origin for an unusually good puzzle book! Most puzzles are accompanied by short and excellent comments which frequently include hints.

Messer, David K., *On the efficiency of a polynomial irreducibility test*, Journal of the ACM 25 (April 1978) 271-282.

This is a nice example of several areas of mathematics overlapping. The basic problem is to decide whether a polynomial is irreducible over the integers mod p for several primes p. To analyze the computing time requirements for this algorithm the author uses a Markov chain model of the testing procedure.

R_{317} is prime, Scientific American 238:2 (February 1978) 89-90.

Hugh C. Williams (Manitoba) together with Eric Seah, an undergraduate, has discovered a new prime number. It has 317 decimal digits, and it is a repunit number--each digit is a 1. It is the first new repunit prime discovered in 50 years. It was singled out by a pseudoprimality test, and further work enabled Williams to show it is prime.

Kilmister, C.W., *Mathematics in the social sciences*, in Duncan, Ronald and Weston-Smith, Miranda (Eds.), The Encyclopedia of Ignorance, Pergamon Pr, 1977; $30, $15 (P), pp. 176-189.

This paper examines the question of why mathematics has not been as successful a tool in the social sciences as it has in physics, and then discusses some new developments that may change the situation. The new developments: looking at aggregates in economy, using simplicial topology to discuss social structures, and (of course) catastrophe theory.

Ferris-Prabhu, A.V., *Forecasting the impossible takes a little longer*, Machine Design (9 February 1978).

Application of Markov chains to forecasting. Specific application is prediction of required stocks of replacement parts for a given machine.

NEWS & LETTERS

1978 U.S.A. MATHEMATICAL OLYMPIAD

The seventh annual U.S.A. Mathematical Olympiad, which took place on May 2, 1978, was prepared under the supervision of Dr. Samuel L. Greitzer.

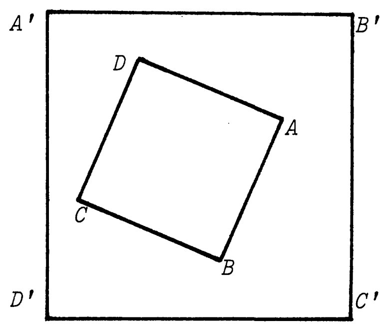
1. Given that a, b, c, d, e are real numbers such that

$$a + b + c + d + e = 8,$$

$$a^2 + b^2 + c^2 + d^2 + e^2 = 16 .$$

Determine the maximum value of e.

2. $ABCD$ and $A'B'C'D'$ are square maps of the same region of a country but drawn to different uniform scales and are superimposed as shown below. Prove that there is only one point O on the small map which lies directly over a point O' of the large map such that O and O' each represent the same place of the country. Also, give a Euclidean construction (straight edge and compass) for O.



3. An integer n will be called *good* if we can write $n = a_1 + a_2 + \ldots + a_k$ where $a_1, a_2, \ldots, a_k$ are positive integers (not necessarily distinct) satisfying $1/a_1 + 1/a_2 + \ldots + 1/a_k = 1$. Given that it is known that the integers 33 through 73 are good, prove that every integer ≥ 33 is good.

4. (a) Prove that if the six dihedral angles (i.e., angles between pairs of faces) of a given tetrahedron are congruent, then the tetrahedron must be regular. (b) Must the tetrahedron be regular if five dihedral angles are congruent?

5. Nine mathematicians meet at an international conference and discover that among any three of them, at least two speak a common language. If each of the mathematicians can speak at most three languages, prove that there are at least three of the mathematicians who can speak the same language.

POSTAGE SEQUENCE

In recent years the price of a first class stamp has risen from 5¢ to 6¢ to 8¢ to 10¢ to 13¢ to 15¢. It seems to be rising to the tune of inflation, but closer analysis reveals the price to be leaping between Fibonacci numbers and triangular numbers. So when you hear about a possible 21¢ rate at some later date don't despair--it will be the voice of someone trying to arrange peace between opposing factions.

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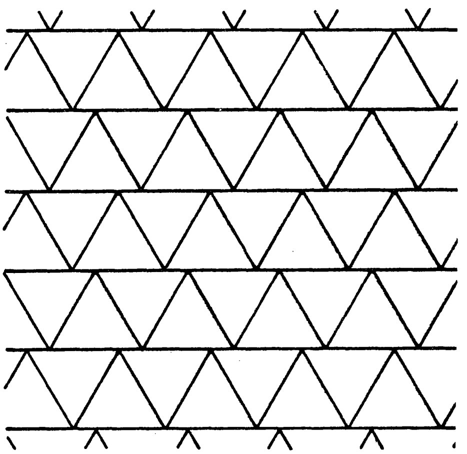
ANOTHER UNIFORM TILING

On page 237 of our article "Tilings by Regular Polygons" (this *Magazine*, November 1977, pp. 227-247), we stated that there exist seven families of uniform tilings by regular polygons that are not edge-to-edge; representatives of the seven families were shown in

Figure 11. Our assertion is incorrect as actually there exist eight such families; the one omitted from the earlier account is shown below:



As with the first three families listed in our article, the tilings of this family use congruent tiles and depend on a real parameter $\alpha (0<\alpha\leq\frac{1}{2})$ which indicates the fraction of overlap between adjacent tiles.

Although it does not excuse our error, the following explanation of how it occurred may be of some interest from a psychological point of view.

In preparing the first draft of our paper we correctly determined all eight families. During rechecking, while preparing a preliminary typed version, we managed to convince ourselves that the tilings in this family were not vertex-transitive; therefore we omitted the family from our list of uniform tilings. The psychologically interesting aspect is that we "checked" and "verified" this incorrect statement several more times since the first submission of the original manuscript; in each case the missing family did turn up as a possible candidate but was at once rejected since we recognized it as "the one that had almost sneaked in earlier!"

It is possible that this type of mistake--the introduction of an error into a manuscript and the failure to rectify it later (even after repeated checks) due to special familiarity with it--occurs quite frequently. It seems likely that many of the mistakes made by previous authors that we mentioned in Section 6 may have arisen in this way. But the most likely candidate for such an explanation is an error committed by Fourrey in *Curiosités géométriques* (Vuibert, Paris 1907; *Fourth Edition*, 1938) which came to our attention since our *Magazine* article was written. In an account of the uniform tilings, Fourrey says that the tiling we denoted by $(3^4.6)$, which is included in his list of candidates and is illustrated by a drawing, is recognized as "failing to be vertex transitive by using the method of tracing it onto a transparent sheet." And this assertion remained unchanged through four editions of *Curiosités géométriques* from 1907 to 1938!

The moral is clear; one should not let down one's guard and allow familiarity and simplicity to excuse one from the careful checking of proofs or assertions in any type of mathematical exposition.

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ADDITIVE SEQUENCES: ERRATUM

There is a technical inaccuracy in my note "Additive Sequences" (this *Magazine*, November 1977, p. 264) which is, fortunately, easy to correct. The passage in question, at the beginning of paragraph 4, refers to the error in the sum of "k terms" of the sum from 2 to infinity of D_i. Now it is certainly true that the sum of the terms up to D_k is $R_k - R_1$, and the next term is of course D_{k+1}. But since the series is defined to start at D_2, the term D_k is actually the $(k-1)$-st term of the series; similarly, the next term D_{k+1} is in fact the k-th term.

In order to leave the major part of the published proof unchanged, the first sentence of paragraph 4 should refer to the correct number $(k-1)$ of terms, for which it is of course necessary to have $k \geq 2$; this must then be established by showing both $N \neq A_1$ and $N \neq A_2$. All of these corrections are easy to establish.

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MULTIPLICATIVE METRIC ON R^n

In her note "A Multiplicative Metric" (this *Magazine*, September 1976, pp. 203-205), Schattschneider asked whether or not her metric on R had a natural extension to R^n. Three readers, as well as the author, responded to this question in the affirmative. In earlier correspondence (this *Magazine*, January 1977, pp. 55-56), R. Johnsonbaugh proposed a coordinate-wise extension of the metric to R^n. Since then, Marlow Sholander, R. Subramanian and K.V. Bhagwat, and Doris Schattschneider have all affirmed that the original metric extends to R^n when Euclidean vector length replaces absolute value in the original definition. Thus, for $x, y, \in R^n$, the function $d(x,y) = |x - y| / (|x| + |y|)$ if x, y are not both 0 and $d(x,y) = 0$ if $x = y = 0$ defines a metric on R^n.

As was the case for R, the only difficult item in the proof is the triangle inequality. All of the invariance properties noted in Schattschneider's note hold for R^n, when suitably interpreted. Thus for $x, y \in R^n$, $n \geq 2$, the following are true:

(i) $d(0,x) = 1$ for all $x \neq 0$, and $d(x,y) = 1$ if x and y have opposite directions as vectors.

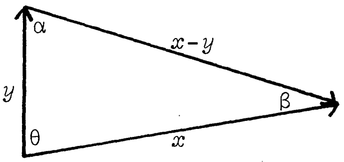
(ii) $0 < d(x,y) < 1$ if and only if $x \neq y$ and either x and y are not collinear or both have the same direction as vectors.

(iii) $d(x,y) = d(ax,ay)$ for all $a \in R$ (for $R^2 = C$, additionally $d(x,y) = d(ax,ay)$ for all $a \in C$).

(iv) $d(x,y) = d(|x|^{-2}x, |y|^{-2}y)$ (for $R^2 = C$, additionally $d(x,y) = d(x^{-1}, y^{-1})$.

(v) $d(x,y) = d(T(x), T(y))$ for any linear isometry T of R^n.

When $x \neq 0$, $y \neq 0$, and $x \neq ty$ for $t \in R$, the vectors x, y, and $x - y$ form a triangle, and $d(x,y)$ is just the length of side $x - y$ divided by the sum of the lengths of the other two sides. Using various familiar identities, and referring to this diagram,



$d^2(x,y)$ can be expressed in many ways:

$$d^2(x,y) = \frac{|x|^2+|y|^2-2|x||y| \cos \theta}{(|x| + |y|)^2}$$

$$= \frac{(|x|-|y|)^2+4|x||y| \sin^2(\theta/2)}{(|x| + |y|)^2}$$

$$= \frac{\sin (\theta/2)}{\cos (\beta/2-\alpha/2)} = \frac{\cos (\beta/2+\alpha/2)}{\cos (\beta/2-\alpha/2)}.$$

To prove the triangle inequality for R^n, it is first shown true for $R^2 = C$. Sholander cleverly uses the invariance properties of d to reduce to a case of specially chosen vectors. His analytic argument employs the first two identities for $d^2(x,y)$ shown above, and uses Minkowski's inequality to complete the proof. Subramanian and Bhagwat offer a geometric proof utilizing the second two identities for $d^2(x,y)$ shown; their proof relies heavily on the use of reductions using trigonometric identities. Both proofs are quite lengthy.

Schattschneider notes that the metric d on $R^2 = C$ is quite similar to the "chordal distance" metric χ associated to the stereographic projection of C onto a sphere tangent to the complex plane which is discussed on pages 42-44 of Hille's *Analytic Function Theory* (Vol. I, 1959, Ginn & Co.). Hille proves that χ is a metric in four lines! He also notes that χ has property (iv), and (v) where T is conjugation. This metric is defined as follows: for $x, y \in C$,

$$\chi(x,y) = \frac{|x-y|}{\sqrt{(1+|x|^2)(1+|y|^2)}}.$$

Since $(1+|x|^2)(1+|y|^2)$

$$= 1 + |x|^2 + |y|^2 + |xy|^2$$

$$\geq |x|^2 + |y|^2 + 2|xy| = (|x|+|y|)^2,$$

it follows that $\chi(x,y) \leq d(x,y)$, with equality holding if and only if $|xy| = 1$. The triangle inequality for d is now easily derived. It is true if any of the vectors involved is 0 (by (i) and (ii)), so suppose that $x, y, z \in C$, and all are non-zero. Let $a = |xy|^{-1/2}$; then $|axay| = 1$. Property (iii) for d and the triangle inequality for χ imply

$$\begin{aligned} d(x,y) &= d(ax,ay) = \chi(ax,ay) \\ &\leq \chi(ax,az) + \chi(az,ay) \\ &\leq d(ax,az) + d(az,ay) \\ &= d(x,y) + d(z,y), \end{aligned}$$

which proves the triangle inequality for d.

The triangle inequality for R^n, $n \geq 3$ is established by reducing it to the case for R^2. Subramanian and Bhagwat use a coordinate-free geometric argument; Sholander's argument, which uses coordinates, is the one we now give. Given $x, y, z \in R^n$, a suitable linear isometry T will transform these vectors into $x = (x_1,0,0,0,..)$, $y = (y_1,y_2,0,0,..)$, and $z = (z_1,z_2,z_3,0,..)$, where x_1, y_2, and z_3 are all non-negative; by property (v) T leaves d invariant. Define $w = (z_1,(z_2^2+z_3^2)^{1/2}, 0,0,..)$. Then $x, y, w \in R^2$, and $d(x,z) = d(x,w)$ and $d(w,y) \leq d(z,y)$. Thus the triangle inequality for R^2 implies $d(x,y) \leq d(x,w) + d(w,y) \leq d(x,z) + d(z,y)$; hence the triangle inequality holds in R^n.

-- *The Editors*

1978 NSF-CBMS REGIONAL RESEARCH CONFERENCES

The National Science Foundation has granted through the Conference Board of the Mathematical Sciences nine Regional Research Conferences for college and university mathematics teachers for the summer and fall of 1978. It is anticipated that three more Regional Conferences will be held later in the 1978-79 academic year.

Date	Host Institution	Subject	Lecturer
May 29-June 2	Rensselaer Polytechnic Institute	The Logic of Computer Programming	Zohar Manna
June 5-9	Shippensburg State College	Multiperson Aspects of Operations Research	W.F. Lucas
June 5-9	SUNY at Buffalo	Integer Programming	E. Johnson
June 5-9	Southern Illinois University	Modeling and Differential Equations in Biological Sciences	H.T. Banks
June 12-16	University of Rhode Island	Approximation Theory (Rational Approximation)	D.J. Newman
July 17-21	Ball State University	Mathematical Developments in Thermodynamics	J.B. Serrin
July 17-21	University of Utah	Spectral and Scattering Theory for Elliptic Operators	S. Agmon
Aug. 21-25	Colorado State University	Non-linear Functional Analysis and Applications	H. Brezis
Oct. 5-9	Oklahoma State University	The Topology of Manifolds	R. Edwards

Approximately 25 mathematicians can attend each conference; travel and subsistence allowances are paid by NSF. Inquiries about particular conferences and requests for application forms should be addressed to the departments of mathematics at the host institutions.

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